Romanian Journal of Mechanics
Volume 4, Issue 2/2019, pp 3-14, ISSN 2537 - 5229, rjm.journals.srmta.ro

DEFECTS IDENTIFICATION IN MICROPOLAR MATERIALS

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Abstract This paper uses the micropolar nonlinear wave theory in order to develop an inverse approach for capturing the size and location of inhomogeneities embedded into a micropolar material. We specify that the inhomogeneities have dimensions comparable to the average grain size of the material. The natural frequencies of a structure represent its signature of the dynamic behaviour, and any defect or change into the internal structure of the material is *felt* by the vibrations in the sense of modifying their natural frequencies. Based on the analysis of the interrelations between natural frequencies and the structure of the material, an unconstrained minimization algorithm is built by minimization of the least square distance between computed and measured natural frequencies. We show that the effect of the size and location of the defects on the natural frequencies of the structures is a real feature that helps us to identify defects into the material.

Key words: Micropolar materials, natural frequencies, inverse problem.

1. INTRODUCTION

The theory of micropolar materials was introduced in the years 1960 by Eringen [1-5]. The theory describes materials with microstructure. The classical continuum mechanics considers that a material particle is characterized by its position only, and no by any orientation within the material. Micropolar material particle has additionally an orientation defined by a *director* which describe the microrotation of the particle [6, 7]. The micropolar material concept is described by the Noll's idea of simple materials originally described for viscous fluids.

The classical continuum mechanics is not inadequate for identification of defects in material. The defects have comparable sizes with the average grain sizes, and the wavelength of the waves is comparable with the average grain size of the granular material and also with the size of defects. That means, the motion (i.e. macro-motion) of the material described by only the deformation functions does not describe movement of grains and of the body constituents even though they may be defects. The micropolar theory undergoes the additional micromotion that describe the rotation and deformation of the constituents at the microscale.

In the deformable body theories, the famous names as Fresnel [8], Navier, Poisson, and Cauchy [9] are known as the precursors to the micropolar theory. Influenced by the Newtonian ideas, Green [10] introduced the continuous systems of points, and Lord Kelvin [11] associated to the Green theory a continuous medium in which a moment may be exerted at any point in the body. Helmholtz [12] and Bertrand [13] developed the Green theory in electromagnetism.

Bernoulli and Euler [9], and also the Poinsot's theory of couples [14] represent a natural way to unite the various concepts of deformable media into a single geometric definition. Starting from the idea that a deformable line is a 1D continuous one-parameter set of triads, a deformable surface is a two-parameter set, results that a deformable medium is a three-parameter set [15]. Let's not forget we can add the time *t* to these geometric descriptions.

This paper advances a method for identifying the defects of a micro-structured plate on the basis of the free vibrations. The effect of shear waves is taken into consideration by considering a third-order theory for the displacement field. There are four type of waves propagating at distinct phase velocities through the plate. But of these, only two coupled shear waves have the wavelengths comparable to the size of defects.

2. BASIC EQUATIONS

The basic equations of the theory of micropolar elasticity are the following [1-6, 16]

Balance of momentum

$$\sigma_{kl,k} - \rho \ddot{u}_l = 0, \tag{1}$$

Balance of moment of momentum

$$m_{rk,r} + e_{klr}\sigma_{lr} - \rho j \ddot{\varphi}_k = 0, \tag{2}$$

Conservation of energy

$$\rho\dot{\varepsilon} = \sigma_{kl}(v_{l,k} - e_{klr}\xi_r) + m_{kl}\xi_{l,k}, \tag{3}$$

Constitutive equations

$$\sigma_{kl} = \bar{\lambda} u_{r,r}\delta_{kl} + \mu(u_{k,l} + u_{l,k}) + \vartheta(u_{l,k} - e_{klr}\varphi_r), \tag{4}$$

$$m_{kl} = \alpha\varphi_{r,r}\delta_{kl} + \beta\varphi_{k,l} + \gamma\varphi_{l,k} \quad , \tag{5}$$

where $C = \{\bar{\lambda}, \mu, \vartheta, \alpha, \beta\}$ are the material moduli, σ_{kl} is stress tensor, ρ is density, u_k is displacement vector, ε is internal energy density, e_{klm} is permutation symbol ($e_{123} = e_{231} = e_{312} = -e_{132} = -e_{321} = -e_{213} = 1$, and all other $e_{klm} = 0$), m_{kl} is couple stress tensor, φ_k is microrotation vector, j is microinertia, v_k is $\dot{u}_k$, ξ_k is $\dot{\varphi}_k$. We use rectangular coordinates x_k ($k = 1, 2, 3$) or $(x_1 = x, x_2 = y, x_3 = z)$.

Indices following a comma indicate partial differentiation, and a superposed dot indicates the time rate. Eringen has shown that

$$0 \le 3\bar{\lambda} + 2\mu + \vartheta, \quad 0 \le \mu, \ 0 \le \vartheta, \quad 0 \le 3\alpha + 2\gamma, \ -\gamma \le \beta \le \gamma \ , \ 0 \le \gamma. \tag{6}$$

Upon substituting (4)-(5) into (1) and (2) we obtain

$$(c_1^2 + c_3^2)\nabla(\nabla u) - (c_2^2 + c_3^2)\nabla\times(\nabla\times u) + c_3^2\nabla\times\varphi = \ddot{u}, \tag{7}$$

where

$$c_c^2 = \frac{\bar{\lambda} + \mu}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}, \quad c_3^2 = \frac{\vartheta}{\rho},$$

$$c_4^2 = \frac{\gamma}{\rho j}, \quad c_5^2 = \frac{\alpha + \beta}{\rho}, \quad \omega_0^2 = \frac{\vartheta}{\rho j} \ . \tag{8}$$

We then decompose the vectors u and φ into scalar and vector potentials $u = \nabla q + \nabla \times U$, $\nabla \cdot U = 0$ and $\varphi = \nabla \varsigma + \nabla \times \phi$, $\nabla \cdot \phi = 0$ which must verify the equations

$$(c_1^2 + c_3^2)\nabla^2 q = \ddot{q}, \quad (c_4^2 + c_5^2)\nabla^2 \varsigma - 2\omega_0^2 \varsigma = \ddot{\varsigma}$$

$$(c_2^2 + c_3^2)\nabla^2 U + c_3^2 \nabla \times \phi = \ddot{U},$$

$$c_4^2 \nabla^2 \phi - 2\omega_0^2 \phi + \omega_0^2 \nabla \times U = \ddot{\phi}. \tag{9}$$

The first two equations (9) are uncoupled, while the last two are a coupled system in vectors U and ϕ.

Consider now that the plane waves propagating in the positive direction of the unit vector n have the form

$$S = S_0 \exp[\mathrm{i}k(n \cdot r - vt)], \tag{10}$$

where and with a, b complex constants A, B complex constant vectors, $k = \dfrac{\omega}{v}$ the wave-number and r the position vector. By substituting (10) into the first two equations (9) we obtain

$$v_1^2 = c_1^2 + c_3^2, \quad v_2^2 = c_4^2 + c_5^2 + 2\frac{\omega_0^2}{k^2}, \tag{11}$$

where v_1 is the velocity of a longitudinal displacement wave and v_2, the velocity of a longitudinal microrotation wave with its microrotation vector in the direction of the propagation. From the last two equations (9) we obtain v_3 and v_4 as roots of the equation

$$\left(1 - 2\frac{\omega_0^2}{\omega^2}\right) X^2 - \left[c_4^2 + c_2^2\left(1 - 2\frac{\omega_0^2}{\omega^2}\right) + c_3^2\left(1 - 2\frac{\omega_0^2}{\omega^2}\right)\right] X + c_4^2(c_2^2 + c_3^2) = 0, \tag{12}$$

with $X = v^2$. The remaining two waves are a transverse displacement wave U of velocity v_3 coupled with a transverse microrotation ϕ of speed v_4. These waves exist only for $\omega \geq \sqrt{2}\omega_0$. Below this frequency waves degenerate into sinusoidal vibrations decaying with distance from the source.

3. DETERMINATION OF THE NATURAL FREQUENCIES RESULTS

Consider a micropolar plate of thickness H, length a and width b. The Cartesian coordinate system $Oxyz$ is located at the middle plane denoted by Ω, with the z axis normal to the plane. A small cube-defect is embedded into the plate and centered at (x_0, y_0, z_0) with size a'. The size of the defect is comparable with the grain size and is characterized by a set of constants $C = \{\bar{\lambda}_0, \mu_0, \vartheta_0, \alpha_0, \beta_0, \gamma_0, \rho_0\}$. Over the whole medium we can write

$$C(x, y) = C^* + C_0(x, y), \tag{13}$$

where

$$C_0(x,y) = \begin{cases} C_0 & \text{for } x,y \text{ belonging to the defect,} \\ 0 & \text{everywhere else.} \end{cases} \tag{14}$$

The both longitudinal q and ς and the transverse U and ϕ waves are taken into consideration. By assuming a harmonic time-dependence, we write

$$U(x,t) = U_0(x)\exp(\mathrm{i}\omega t) \quad , \quad \phi(x,t) = \phi_0(x)\exp(\mathrm{i}\omega t), \tag{15}$$

with $U_0(x)$ and $\phi_0(x)$, the unknown functions.

The shear deformation theory proposed by Frederiksen [17] is based on a displacement field in which the displacement in the x and y directions are expanded as cubic functions of the thickness coordinate, and the transverse deflection is assumed to be constant through the thickness. For a micropolar elastic body we extend this theory by supposing that $S = \{U_0, \phi_0\}$ are cubic functions of the thickness coordinate

$$S_i(x,y,z) = z\Psi_i(x,y) + z^2\Phi_i(x,y), \tag{16}$$

with $i = 1,2$. In our notation $S_1 = U_0$ and $S_2 = \Phi_0$.

The unknown expansion functions are Ψ, Φ, Σ . Following the Ritz procedure, we assume the solution for Ψ, Φ, Σ are finite series with unknown coefficients

$$\Psi_i(x,y) = \sum_{m,n}^{N} X_{imn} w_m\left(\frac{2x}{a}\right) w_n\left(\frac{2y}{b}\right),$$

$$\Phi_i(x,y) = \sum_{m,n}^{N} Y_{imn} w_m\left(\frac{2x}{a}\right) w_n\left(\frac{2y}{b}\right) \ , \tag{17}$$

$$\Sigma_i(x,y) = \sum_{m,n}^{N} Z_{imn} w_m\left(\frac{2x}{a}\right) w_n\left(\frac{2y}{b}\right), \ i = 1,2,3,4$$

where $\sum_{m.n}^{N} = \sum_{m=0}^{p-1} \sum_{n=0}^{p-1}$ and $w_m(\varepsilon)$, $-1 \le \varepsilon \le 1$ are the assumed functions defined for the nondimensional variable ε.

These functions are chosen to satisfy the following requirements: both functions and their first derivatives are continuous; the functions are complete and admissible i.e. satisfy the boundary conditions of the plate. A good set of functions is the one of the degenerated beam functions [17]

$$w_0(\varepsilon) = 1 \ , \quad w_1(\varepsilon) = \varepsilon$$

$$w_m = \cos k_I \varepsilon, \ I = \frac{m+2}{2}, \ m = 2,6,10,...$$

$$w_m = \cosh k_J \varepsilon, \ J = \frac{m+1}{2}, \quad m = 3,7,11,... \tag{18}$$

$$w_m = \sin k_I \varepsilon, \quad I = \frac{m+2}{2}, \quad m = 4,8,12,...$$

$$w_m = \sinh k_J \varepsilon, \ J = \frac{m+1}{2} \ , \quad m = 5,9,13,...$$

where k_m are the solution of the equation

$$\tan k_m + (-1)^m \tanh k_m = 0 \quad , \quad m = 2,3,4,... \tag{19}$$

Though the functions (18) do not satisfy free edge conditions, it was shown [17] that for problems involving free edges the series composed of these functions converge rapidly towards highly accurate solutions. The arbitrary coefficients in the series (17) are determined by minimizing a functional defined as to measure how well equations (9) are verified

$$F = \int_{\Omega} \int_{-H/2}^{H/2} \sum_{i=1}^{4} s_i^2 \,, \tag{20}$$

where

$$\begin{aligned} s_1 &= (c_1^2 + c_3^2)\nabla^2 q + \omega^2 q \,, \\ s_2 &= (c_4^2 + c_5^2)\nabla^2 \varsigma - 2\omega_0^2 \varsigma + \omega^2 \varsigma \,, \\ s_3 &= (c_2^2 + c_3^2)\nabla^2 U + c_3^2 \nabla \times \phi + \omega^2 U \,, \\ s_4 &= c_4^2 \nabla^2 \phi - 2\omega_0^2 \phi + \omega_0^2 \nabla \times U + \omega^2 \phi \,. \end{aligned} \tag{21}$$

The procedure yields to an eigenvalue problem with ω^2 as the eigenvalue and the unknown coefficients as the eigenvector.

4. INVERSE PROBLEM

Suppose that the defect is described by the following parameters: its centre (x_0, y_0, z_0), side a' and the set of parameters $\{\bar{\lambda}_0, \mu_0, \rho_0\}$ which are different from the matrix parameters.

$$P = (x_0, y_0, z_0, a', \bar{\lambda}_0, \mu_0, \rho_0) \,. \tag{22}$$

The objective of the optimisation procedure is to minimise the difference between $Q^{mes}(P)$ and $Q^{calc}(P)$ where $Q = \{\omega_1, \omega_2, ..., \omega_N\}$. Q^{mes} represents the measured natural frequencies of the plate and Q^{calc} the corresponding calculated values as a function of the parameters P. Thus, the objective function can be formulated in the sense of the global normalised least squares error

$$\Psi(P) = \frac{\sum_{j=1}^{N} (Q_j^{mes} - Q_j^{calc})^2}{\sum_{j=1}^{N} (Q_j^{mes})^2} \,. \tag{23}$$

The Davidon-Fletcher-Powell quasi-Newton algorithm has been used because it requires a relatively low number of objective function evaluations and because of its ability to converge quickly near minima.

As an example, we consider a polycrystalline metal plate whose grain size is approximately 0.5×10^{-5} m. In order to simplify the numerical analysis we suppose that this material is characterized by

$$\bar{\lambda} = \mu = 40\text{GPa}\,, \quad \vartheta = 0.2\text{GPa}\,, \quad \alpha = \beta = \gamma = 3\text{GN}\,, \quad j = 6.25 \times 10^{-7}\,\text{m}^2\,, \quad \rho = 1160\,\text{kg/m}^3 \,.$$

The plate dimensions are

$$a = 100\text{mm}, \ b = 110\text{mm}, \ H = 2\text{mm}.$$

We consider a cubic defect of the side $a' = 2\times10^{-5}\text{m}$, located in the point $(15,-15,0)$ characterized by

$$\overline{\lambda}_0 = 4\times\lambda, \ \mu_0 = 4\times\mu, \ \vartheta_0 = \vartheta, \ \alpha_0 = \alpha, \ \beta_0 = \beta, \ \gamma_0 = \gamma \ , \ \rho_0 = 1.5\times\rho \ .$$

First, we have applied the direct problem to calculate the natural frequencies. The dispersion curves for both longitudial and transversal waves are shown in Fig.1.

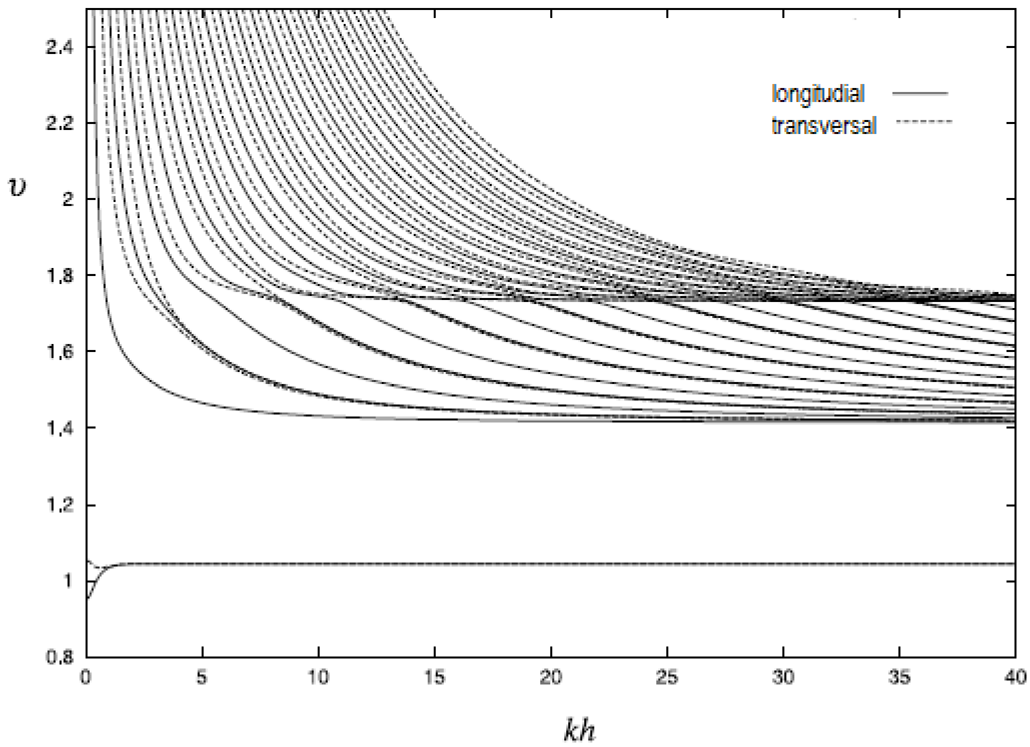


Fig.1. Dispersion curves for both longitudinal and transversal waves.

The convergence is very rapidly convergence achieved when degenerated beam functions (18)-(19) are employed.

To calculate the eigenvalues, the Podlevskii procedure is used [18, 19]. Computational aspects of this procedure are presented next.

Podlevskii uses an iterative algorithm for finding the maximum and minimum approximations for each eigenvalue λ. The algorithm is based on a numerical procedure for calculating the first and second derivatives of the determinant of the problem. It is well known that, for any fixed λ, the matrix $D(\lambda)$ is

$$D(\lambda) = L(\lambda)U(\lambda), \tag{24}$$

where $L(\lambda)$ is a lower triangular matrix with unit diagonal and $U(\lambda)$ is an upper triangular matrix.

We add that the characteristic equation

$$f(\lambda) = \det D(\lambda), \tag{25}$$

is analytic. We suppose that f has m zeros $\lambda_1, \lambda_2, ..., \lambda_m$, where the number m is given by

$$m = s_0 = \frac{1}{2\pi \mathrm{i}} \int \frac{f'(\lambda)}{f(\lambda)} \mathrm{d}\lambda \,. \tag{26}$$

Defining

$$s_k = \frac{1}{2\pi \mathrm{i}} \int \lambda^k \frac{f'(\lambda)}{f(\lambda)} \mathrm{d}\lambda \,, \quad k = 1, 2, ..., m \,, \tag{27}$$

we have

$$\sum_{j=1}^{m} \lambda_j^k = s_k \,, \quad k = 1, 2, ..., m \,. \tag{28}$$

Podlevskii procedure uses an iterative two-sided algorithm for refining the rough approximations to the eigenvalues that were obtained by using some or other method.

Suppose that λ_0 is an approximation of the eigenvalue λ^*. The iteration procedure can be initiated from λ_0

$$\lambda_{2m+1} = \lambda_{2m} - \frac{f(\lambda_{2m}) f'(\lambda_{2m})}{f'(\lambda_{2m})^2 - f(\lambda_{2m}) f''(\lambda_{2m})} \,, \quad m = 0, 1, ... \tag{29}$$

$$\lambda_{2m+2} = \lambda_{2m+1} - \frac{f(\lambda_{2m+1})}{f'(\lambda_{2m+1})} \,.$$

If the initial approximation is situated to the left of λ^*, then the others approximations of λ^* are calculated as

$$\lambda_0 < \lambda_2 < ... < \lambda_{2m} < ... < \lambda^* < ... < \lambda_{2m-1} < ... < \lambda_3 < \lambda_1 \,, \tag{30}$$

or

$$\lambda_0 < \lambda_1 < ... < \lambda_{2m-1} < ... < \lambda^* < ... < \lambda_{2m} < ... < \lambda_4 < \lambda_2 \,. \tag{32}$$

If the initial approximation is situated to the right of λ^*, then the others approximations of λ^* are calculated as

$$\lambda_1 < \lambda_3 < ... < \lambda_{2m-1} < ... < \lambda^* < ... < \lambda_{2m} < ... < \lambda_2 < \lambda_0 \,, \tag{30}$$

or

$$\lambda_2 < \lambda_4 < ... < \lambda_{2m} < ... < \lambda^* < ... < \lambda_{2m-1} < ... < \lambda_1 < \lambda_0 \,. \tag{32}$$

In the iterative process (29), the values of $f(\lambda)$ and its derivatives at a specific λ are used by using of decomposition

$$\begin{aligned} D &= LU, \\ B &= MU + LV, \\ C &= NU + 2MV + LW. \end{aligned} \tag{33}$$

The first 12 natural frequencies were obtained from the Podlevskii procedure

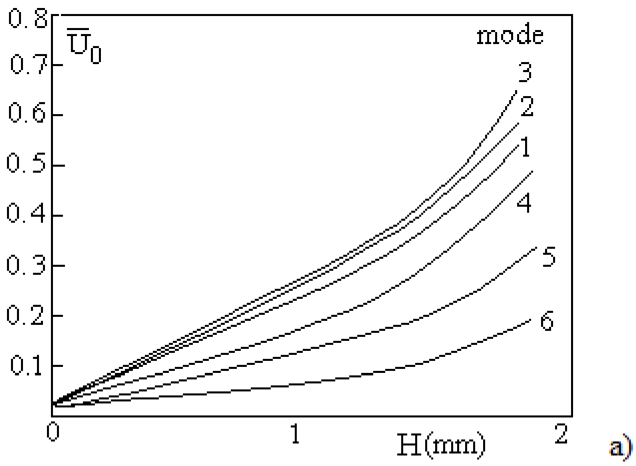
115.33, 365, 65, 432,97, 489.11, 543.3, 774.92,

995.87, 1076.44, 1320.4, 1341.8, 1456.4, 1559.2 Hz

The two-sided approximations of the first 5 the eigenvalues are presented in Table 1

Table 1. Two-sided approximations of the natural frequencies (Hz).

frequency	Left side	Right side
1	114.13	115.98
2	361, 05	367, 44
3	430,22	433,17
4	487.10	490.01
5	542.99	544.31
6	770.42	775.33
7	993.27	997.47
8	1075.04	1077.15
9	1318.40	1322.11
10	1340.12	1343.08
11	1455.40	1458.13
12	1557.02	1560.42



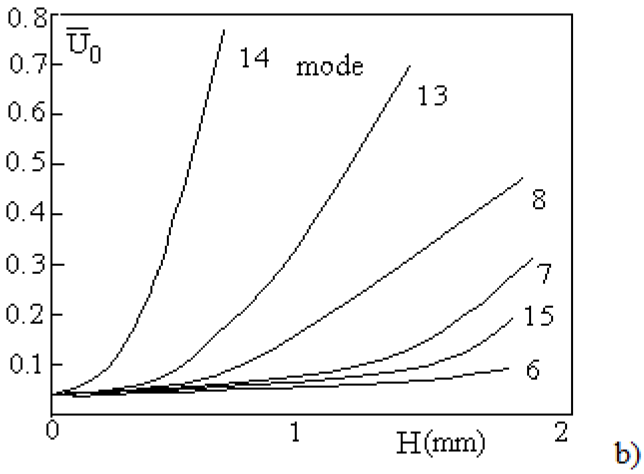


Fig.2. Influence of dimensionless displacement amplitude on natural frequencies for different modes.

We have no experimental results for the problem, so we built so-called experimental measurements for material by using an easy perturbation of the frequencies calculated with Podlevskii procedure.

So, 18 *measured natural frequencies* are used for solving the inverse problem

114.33, 365.65, 432.97, 489.11, 543.3, 774.92, 995.87, 1076.44, 1320.4, 1341.8, 1456.4, 1559.2, 1678, 1805.4, 1988.3, 2290.45, 2654, 2876 Hz

The results obtained by applying a genetic algorithm are

$$x_0 = 15.3\,,\ \ y_0 = -15.2\,,\ \ z_0 = 0.2\,,\ \ a' = 2.9\times 10^{-5}\,\mathrm{m}\,,$$

$$\bar{\lambda}_0 = 4.19\times\bar{\lambda}\,,\ \ \mu_0 = 3.87\times\mu\,,\ \ \vartheta_0 = \vartheta\,,\ \ \alpha_0 = \alpha\,,\ \ \beta_0 = \beta\,,\ \ \gamma_0 = \gamma\ \ ,\ \ \rho_0 = 1.56\times\rho\ \ .$$

Figs. 2 and 3 give the influence of shear waves (dimensionless quantities $\bar{U}_0$ and $\bar{\Phi}_0$) on the natural frequencies for different modes. Mode 3 of the micropolar plate without/with defect is displayed in Fig.4.

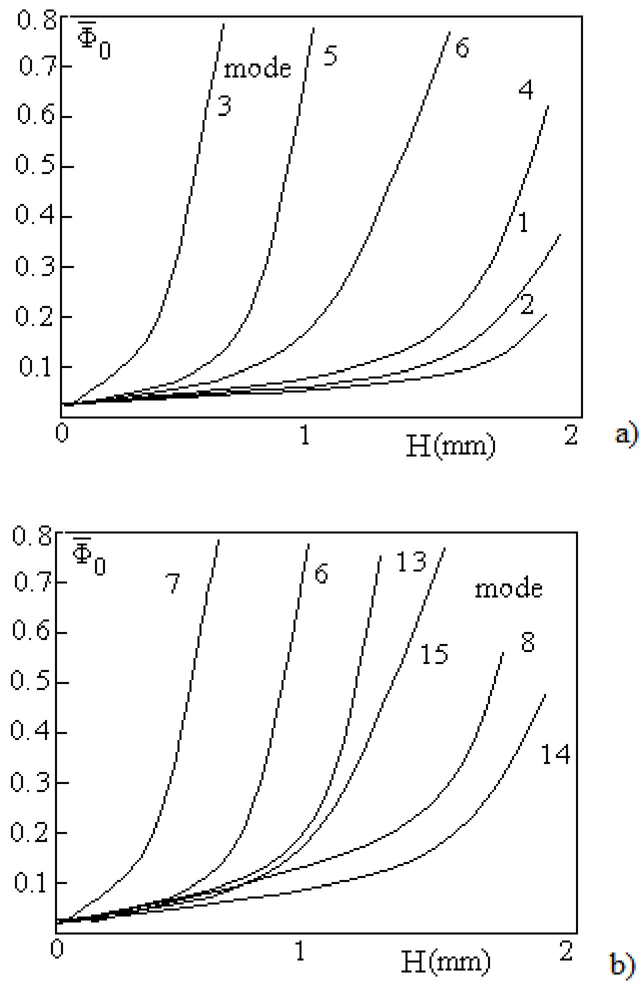


Fig.3. Influence of dimensionless microrotation on natural frequencies for different modes.

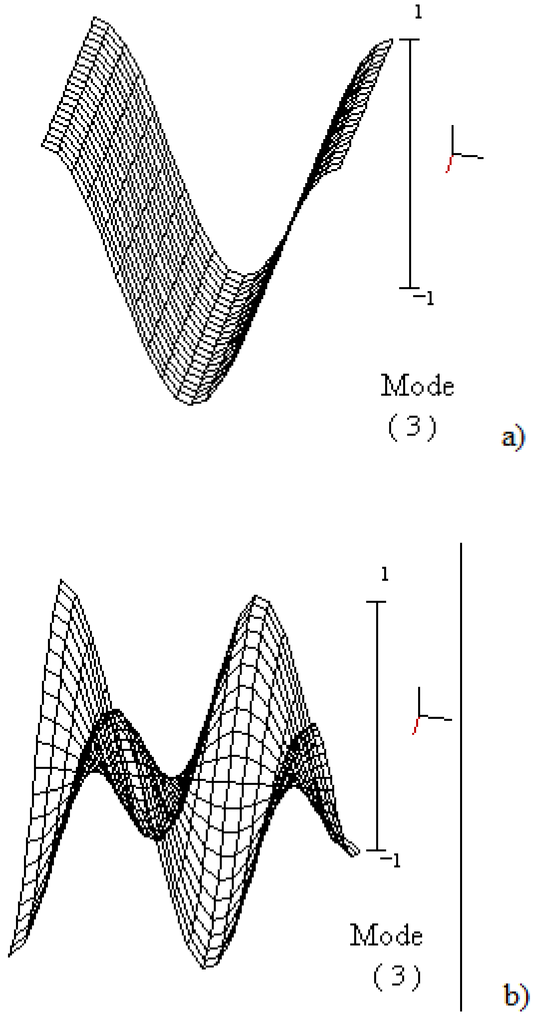


Fig.4. Mode 3 for micropolar plate without/with defect.

The displacement around the cubic defect are illustrated via contour plots (Fig. 5). Due to the anisotropy of the defect, the displacement is strongly angle-dependent.

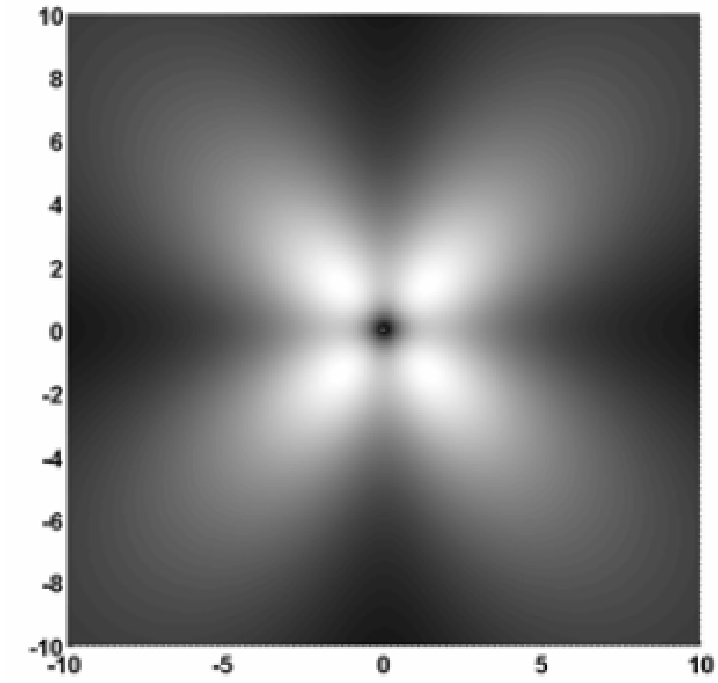


Fig.5. Displacement field in the vicinity of the cubic defect via the contour plotting.

As we said before, the natural frequencies of a material represent its signature of the dynamic behaviour, and any defect or change into the internal structure of the material is *felt* by the vibrations in the sense of modifying their natural frequencies and also the vibration modes.

5. **CONCLUSIONS**

This paper shortly presents the micropolar nonlinear wave theory in order to develop an inverse approach for detecting the size and location of inhomogeneities embedded into a micropolar material. Based on the analysis of the interrelations between natural frequencies and the structure of the material, an inverse problem is built by minimization of the least square distance between computed and measured natural frequencies.

To determine the natural frequencies, an iterative algorithm developed by Podlevskii, is used. In this way, the finding of two-sided approximations to the eigenvalues are performed. The algorithm uses a numerical procedure for calculating the first and second derivatives of the determinant.

The inverse problem is based on a higher-order shear wave theory which accounts for parabolic distribution of the transversal shear amplitudes of the displacements and microrotations waves through the thickness of the plate. Due to the dimension of defects comparable with the average grain size, the shear displacements waves do not provide sufficient information for a correct solving of the inverse problem.

The identification of defects is fast and simple to perform when both shear displacements and shear microrotation waves are used.

The conclusion is that the influence of shear waves on the natural frequencies is very important for micropolar materials.

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Received September 20, 2019