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ON THE COULOMB VIBRATIONS IN THE PENDULUM MOTION

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Abstract In this paper, the Coulomb vibrations are discussed in order to describe the motion of the double pendulum. The pendulum`s interest is associated to mechanical clocks, the metronomes and seismometers. The linear equivalence method (LEM) formulated by Toma (1995) is applied to define the Coulomb vibration as a particular solution of the nonlinear system of equations that describes the pendulum motion. The general solutions are written as a linear superposition of Coulomb vibrations.

Key words: Double pendulum, Coulomb vibrations, LEM.

1. INTRODUCTION

The vibration behavior of the double pendulum is rich and complex [1, 2]. The most application of pendulums is the clock. The first pendulum clock built in the 1600s was the most accurate clock for nearly 300 years. A pendulum inside the clock keeps the hands running on time, since the motion of a pendulum is a constant time interval. Seismometers and metronomes also use the pendulum to measure the seismic activity in the ground or to reading the music.

The first pendulum was found in a seismometer in the time of the Han Dynasty. The pendulum work consists in activating a series of levers that directed a small ball to fall out of one of the instrument's eight holes. The metronome emits a sound for each beat of a certain interval with the help of a pendulum [3].

In this paper, the capability of the linear equivalence method is extended to obtain the analytical representations of the non-simple-regular solutions of pendulum describable as a linear superposition of Coulomb vibrations [4, 5]. The Coulomb vibration is an elementary function that describes a vibration unit used for developing the general solution. The solutions are designed to establish some qualitative conclusions, relevant for describing the correct dynamical behavior of the pendulum. For large motions it is a chaotic system, but for small motions it is a simple linear system.

The method we have used is the linear equivalence method (LEM) formulated by Toma [6, 7]. The LEM is applicable for the integration of first-order differential nonlinear differential equations having algebraic nonlinearities and an arbitrary number of unknowns. This method becomes consistently in capturing qualitatively and quantitatively the contribution of all nonlinear terms and yielding highly accurate solutions.

2. EQUATIONS OF MOTION

Fig. 1 shows a double pendulum consisted from two straight rods O_1O_2 and O_3O_4 of masses M_1 , M_2, lengths $2l_1$, $2l_2$, and mass centres C_1, C_2 [4, 8, 9-11]. The rods are articulated in O_3 and suspended in O_1, so that they can move in the vertical plane xO_1y without friction. Other notations from fig. 1 are: $l = O_1O_3$, $l_1 = O_1C_1$, $l_2 = O_3C_2$. We note by θ_1 and θ_2 the displacement angles in rapport to the vertical O_1x (degree of freedom), I_1 the mass moment of inertia of O_1O_2 with respect to C_1, I_2 the mass moment of inertia of O_3O_4 with respect to C_2, and g the gravitational constant. The forces acting upon the pendulum are the weights of bars. The generalised forces are

$$G_1 = -M_1 l_1 g \sin\theta_1 - M_2 g l \sin\theta_1,$$
$$G_2 = -M_2 g l_2 \sin\theta_2 \ . \tag{1}$$

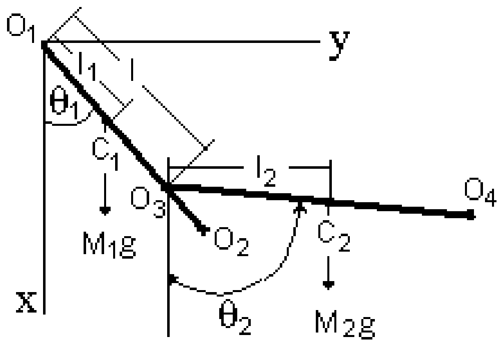

Fig.1. Geometry of the system.

The motion equations obtained from the Lagrange equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} &= G_1, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} &= G_2, \end{aligned} \tag{2}$$

are dimensionless quantities in the form

$$\begin{cases} \ddot{\theta}_1 + \alpha[\ddot{\theta}_2 \cos(\theta_2 - \theta_1) - \dot{\theta}_2^2 \sin(\theta_2 - \theta_1)] + \beta \sin\theta_1 = 0, \\ \ddot{\theta}_2 + \gamma[\ddot{\theta}_1 \cos(\theta_2 - \theta_1) + \dot{\theta}_1^2 \sin(\theta_2 - \theta_1)] + \sin\theta_2 = 0, \end{cases} \tag{3}$$

where dimensionless variables and coefficients are given by

$$t \to t\sqrt{\eta}, \quad \eta = \frac{g M_2 l_2}{I_2 + l_2^2 M_2}, \tag{4}$$

$$\alpha = \frac{M_2 l l_2}{I_1 + l_1^2 M_1 + l^2 M_2}, \quad \gamma = \frac{M_2 l l_2}{I_2 + l_2^2 M_2},$$
$$\beta = \frac{M_1 l_1 + M_2 l}{I_1 + l_1^2 M_1 + l^2 M_2} \cdot \frac{I_2 + l_2^2 M_2}{M_2 l_2}.$$

The dot represents the differentiation with respect to the nondimensional variable *t*. By setting

$$m = \frac{M_1}{M_2}, r = \frac{l}{l_1}, s = \frac{l_2}{l_1}, \tag{5}$$

the motion of the double pendulum depends on the control parameters r, s and m. It is simple to show that α, β, γ become

$$\alpha = \frac{rs}{r^2 + \frac{4}{3}m}, \ \beta = \frac{4}{3} s \frac{r+m}{r^2 + \frac{4}{3}m}, \ \gamma = \frac{3}{4}\frac{r}{s}, \tag{6}$$

By introducing the new variables

$$\theta_1 = z_1, \quad \theta_2 = z_2, \ \dot{\theta}_1 = z_3, \quad \dot{\theta}_2 = z_4, \tag{7}$$

equations (3) are rewritten in the state-space form as

$$\begin{aligned}
\dot{z}_1 &= z_3, \\
\dot{z}_2 &= z_4, \\
\dot{z}_3 &= \frac{1}{1-\alpha\gamma\cos^2(z_2 - z_1)}[-\beta \sin z_1 + \alpha \sin z_2 \cos(z_2 - z_1) + \\
&\quad + \alpha z_4^2 \sin(z_2 - z_1) + \alpha\gamma z_3^2 \sin(z_2 - z_1)\cdot\cos(z_2 - z_1)], \\
\dot{z}_4 &= \frac{1}{1-\alpha\gamma\cos^2(z_2 - z_1)}[-\sin z_2 + \beta\gamma \sin z_1 \cos(z_2 - z_1) - \\
&\quad - \gamma z_3^2 \sin(z_2 - z_1) - \alpha\gamma z_4^2 \sin(z_2 - z_1)\cos(z_2 - z_1)],
\end{aligned} \tag{8}$$

with $1-\alpha\gamma\cos^2(z_2 - z_1) \neq 0$.

The system of equations (8) is rewritten in the form

$$\dot{z}_n = A_{np} z_p + \frac{g_n(z)}{h_n(z)}, \quad n = 1,2,3,4, \tag{9}$$

with

$$\begin{aligned}
g_n(z,t) &= B_{np} z_p + C_{np} \sin z_p + D_{npq} \cos(z_p - z_q)\sin z_p + \\
&\quad + E_{npqr} z_p \ \cos(z_q - z_r) + F_{npqr} z_p^2 \sin(z_q - z_r) + \\
&\quad + G_{npqr} z_p^2 \sin(z_q - z_r)\cos(z_q - z_r) - A_n(z) + \\
&\quad + L_{nnpqm} A_m(z)\cos(z_p - z_q),
\end{aligned} \tag{10}$$
$$h_n(z) = 1 - H_{npq}\cos^2(z_p - z_q),$$

and $h_n \neq 0$. It is assumed to be valid the summation law with respect to repeated indices $(n, p, q, r = 1,2,3,4)$. The constants are

$$\begin{aligned}
&A_{13} = 1, \quad A_{24} = 1\ , \quad A_{56} = 1\ ,\\
&B_{45} = 1, \quad C_{31} = -\beta, \quad C_{42} = -1,\\
&D_{412} = \beta\gamma\ , \ D_{321} = \alpha,\\
&E_{3521} = -\alpha,\\
&F_{3421} = \alpha, \quad F_{4321} = -\gamma,\\
&G_{3321} = \alpha\gamma, \quad G_{4421} = -\alpha\gamma,\\
&H_{321} = \alpha\gamma, \quad H_{421} = \alpha\gamma,\\
&L_{3214} = \alpha, \quad L_{4214} = \gamma,
\end{aligned}$$

and the rest are zero.

The initial conditions are

$$z_1(0) = z_{10}, \quad z_2(0) = z_{20}, \quad z_3(0) = z_{30}, \quad z_4(0) = z_{40}. \tag{11}$$

The linearized form of equations (2.12) is

$$\dot{z} = Az, \tag{12}$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta\varsigma & \alpha\varsigma & 0 & 0 \\ \beta\gamma\varsigma & -\varsigma & 0 & 0 \end{bmatrix}, \tag{13}$$

where $\varsigma = 1 + \alpha\gamma$. The characteristic equation $|A - \lambda I| = 0$

$$[\lambda^4 + \varsigma(\beta+1)\lambda^2 + \beta\varsigma^2(1-\alpha\gamma)] = 0, \tag{14}$$

admits the roots $\pm ip_1, \ \pm ip_2$.

3. THE LINEAR EQUIVALENCE METHOD (LEM)

We analytically solve the nonlinear system of equations (9) and (11) by introducing the linear equivalence transformation (LEM) that depends on four parameters $\sigma_i \in R$, $i = 1,2,3,4$ [6-9]

$$v(t,\sigma) = \exp\Big[\sum_{j=1}^{4} \sigma_j z_j\Big], \quad \sigma_i \in R, \tag{15}$$

with the associated initial conditions

$$v(0,\sigma) = \exp\Big[\sum_{j=1}^{4} \sigma_j z_j^0\Big]. \tag{16}$$

Using (16), the system (15) is transformed into a nonlinear partial differential equation of the first order

$$\frac{\partial v}{\partial t} - D(t,\sigma,v) = 0\,, \tag{17}$$

where $D(t,\sigma,v)$ is the formal nonlinear differential operator associated to (9).

We look for a solution $v(t,\sigma)$ of (17) under the form

$$\begin{aligned} v(t,\sigma) = 1 &+ \sum_{k=1}^{4} \sum_{i=1} A_k^i \frac{\sigma_k^i}{i!} + \sum_{\substack{k,l=1\\k\neq l}}^{6} \sum_{i,j=1} A_k^i A_l^j \frac{\sigma_k^i \sigma_l^j}{i!\,j!} + \\ &+ \sum_{\substack{k,l,m=1\\k\neq l\neq m}}^{4} \sum_{i,j,r=1} A_k^i A_l^j A_r^r \frac{\sigma_k^i \sigma_l^j \sigma_m^r}{i!\,j!\,r!} + \\ &+ \sum_{\substack{k,l,m,n=1\\k\neq l\neq m\neq n}}^{4} \sum_{i,j,r,s=1} A_k^i A_l^j A_r^r A_n^s \frac{\sigma_k^i \sigma_l^j \sigma_m^r \sigma_n^s}{i!\,j!\,r!\,s!}, \end{aligned} \tag{18}$$

and introduce it into (17). By using the series expansions for circular functions, the unknown functions $A_n(t)$, $n = 1,2,3,...,6$, are determined by equating the terms of the same power in σ and t. So, we obtain

$$A_n(t) = \sum_{k,\eta=0} \{(\mu t)^{k+1} \tilde{A}_{nk}(\eta) \;\; \Phi_k(\mu t,\eta) + \;\; (\mu t)^k \tilde{B}_{nk}(\eta)\psi_k(\mu t,\eta)\}\,, \tag{19}$$

and $k = 0,1,2,3,...k_{\max}$, $\eta = 0,1,2,3,...$

We determine, via numerical experimentation, which $k_{\max}$ (depending of η) is relevant for capturing the contribution of all nonlinearities of governing equations. So, we assess the efficiency of the method for $k_{\max} = 7$. For $k_{\max} > 7$ we do not have significant terms in solutions. In (19) the functions $\Phi_k(\mu t,\eta)$ and $\psi_k(\mu t,\eta)$ are given by

$$\begin{aligned} \Phi_k(\mu t,\eta) &= \sum_{m=k+1} (\mu t)^{m-k-1} A_m^{(k)}(\eta)\,, \\ \psi_k(\mu t,\eta) &= \sum_{m=k+1} (\mu t)^{m-k-1} B_m^{(k)}(\eta)\,, \end{aligned} \tag{20}$$

and $\mu = \mu(\eta)$

$$\mu = \sum_{j=1}^{4} \alpha_j \lambda_j\,, \tag{21}$$

with $\sum_{l=1}^{4} \alpha_j = \eta + 1, \quad \alpha_j \geq 0, \quad j = 1,2,3,4$.

The quantities λ_j in (21) are $\lambda_1 = p_1$, $\lambda_2 = -p_1$, $\lambda_3 = p_2$, $\lambda_4 = -p_2$, where $\pm ip_1$, $\pm ip_2$ are the roots of the characteristic equation (14).

The unknowns $\tilde{A}_{nk}$, $\tilde{B}_{nk}$ in (19) are expressed by

$$\begin{aligned} \tilde{A}_{nk}(\eta) &= C_k(\eta) B_{nk}(\eta), \\ \tilde{B}_{nk}(\eta) &= C_k(\eta) C_{nk}(\eta), \end{aligned} \tag{22}$$

$$C_k(\eta) = \frac{\sqrt{k^2+\eta^2}}{k(2k+1)} C_{k-1}(\eta), \tag{23}$$

$$C_0(\eta) = \frac{p\,|\,\Gamma(1+i\eta)\,|}{2!}, \tag{24}$$

where Γ is the gamma function and

$$\left|\frac{\Gamma(1+i\eta)}{\Gamma(2)}\right|^2 = \prod_{n=0}[1+\frac{\eta^2}{(2+n)^2}]^{-1}. \tag{25}$$

The constants $A_m^{(k)}$, $B_m^{(k)}$ are related by the relation

$$B_m^{(k)}(\eta) = m\Delta A_m^{(k)}(\eta), \tag{26}$$

$$\begin{aligned} &A_{k+1}^{(k)} = 1, \quad A_{k+2}^{(k)} = \frac{\eta}{k+1}, \\ &(m+k)(m-k-1)A_m^{(k)} = 2\eta A_{m-1}^{(k)} - A_{m-2}^{(k)}. \qquad m > k+2, \end{aligned} \tag{27}$$

The constants $B_{nk}(\eta)$, $C_{nk}(\eta)$ depend on initial conditions and on coefficients from governing equations. From (3.6) we obtain

$$\psi_k(\mu t, \eta) = \sum_{m=k+1} (\mu t)^{m-k-1} B_m^{(k)}(\eta). \tag{28}$$

The function $\psi(\mu t, \eta)$ is linked to the $\Phi(\mu t, \eta)$ by

$$\Psi(\mu t, \eta) = \mu \frac{d}{dt}\Phi(\mu t, \eta) = \mu\dot{\Phi}(\mu t, \eta). \tag{29}$$

By noting

$$F_k(\mu t, \eta) = C_k(\eta) \quad (\mu t)^{k+1}\Phi_k(\mu t, \eta), \tag{30}$$

the representation (19) becomes

$$A_n(t) = \sum_{\eta,k=0} \{B_{nk}(\eta)F_k(\mu t, \eta) + C_{nk}(\eta)\dot{F}_k(\mu t, \eta)\}], \tag{31}$$

where the functions $F_k(\mu t, \eta)$ are the elementary Coulomb functions [10-12].

These functions represent the elementary units of vibrations used in this paper to develop the general solutions of the pendulum motion. We have

$$F_0(\mu t, 0) = \sin\mu t, \quad \dot{F}_0(\mu t, 0) = \cos\mu t. \tag{32}$$

So, the representation of the solution of the nonlinear system of equations (9) with the initial conditions (11) is found to be

$$z_{n(t)} = A_n(t). \tag{33}$$

As mentioned above, the solutions of nonlinear equations that govern the motion of a double pendulum are written as a linear superposition of Coulomb vibrations.

The first terms in (33) representing the linear part of the solution of (13) in the case of small oscillations ($n = 1, 2, \quad k = 0, 1, \ \eta = 0$) are

$$z_1 = B_{10}(0)\sin p_1 t + C_{10}(0)\cos p_1 t + B_{11}(0)\sin p_2 t + C_{11}(0)\cos p_2 t\,,$$

$$z_2 = B_{20}(0)\sin p_1 t + C_{20}(0)\cos p_1 t + B_{21}(0)\sin p_2 t + C_{21}(0)\cos p_2 t\,, \tag{34}$$

$$B_{10} = B_{20}(1-\frac{p_1^2}{\gamma})\,,\;\; C_{10} = C_{20}(1-\frac{p_2^2}{\gamma})\,,$$

$$B_{11} = B_{21}(1-\frac{p_1^2}{\gamma})\,,\;\; C_{11} = C_{21}(1-\frac{p_2^2}{\gamma})\,. \tag{35}$$

where p_1, p_2 are roots of the characteristic equation (14), and the constants B_{20}, B_{21}, C_{20} and C_{21} are determined from initial conditions.

For $\eta = 0$ and $k > 2$ in the solutions appear additional terms of the form $\sin(ap_1 + bp_2)t$, $\cos(ap_1 + bp_2)t$ with $a + b = 1,2,3...$ For $\eta \neq 0$ the solutions will contain the functions $F[(ap_1 + bp_2)t, \eta]$ with $a + b = 1,2,3...$

3. NUMERICAL INVESTIGATIONS

We assess the efficiency of our analysis in computing the representations of solutions for the double-pendulum. Numerical experiments on the solutions have shown that for the motion of pendulum is bounded and stable when initial conditions are chosen in the interval [-1.5, 1.5] .

The double pendulum shows a sensitive dependence on initial conditions for the interval [-3, 3].

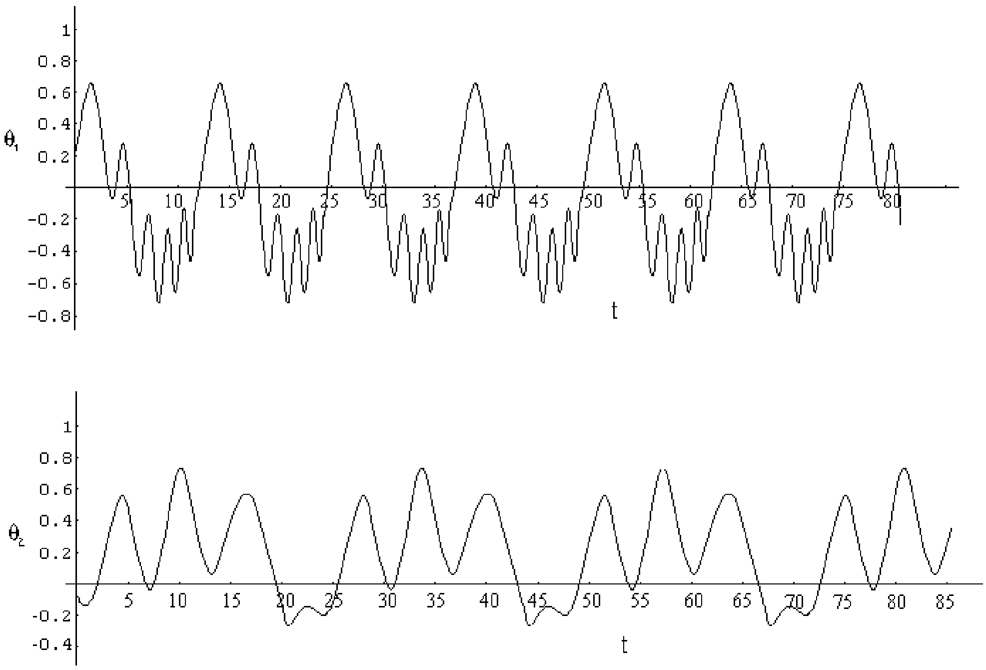

Fig.1. Solutions θ_1(t) and θ_2 (t) for Example 1.

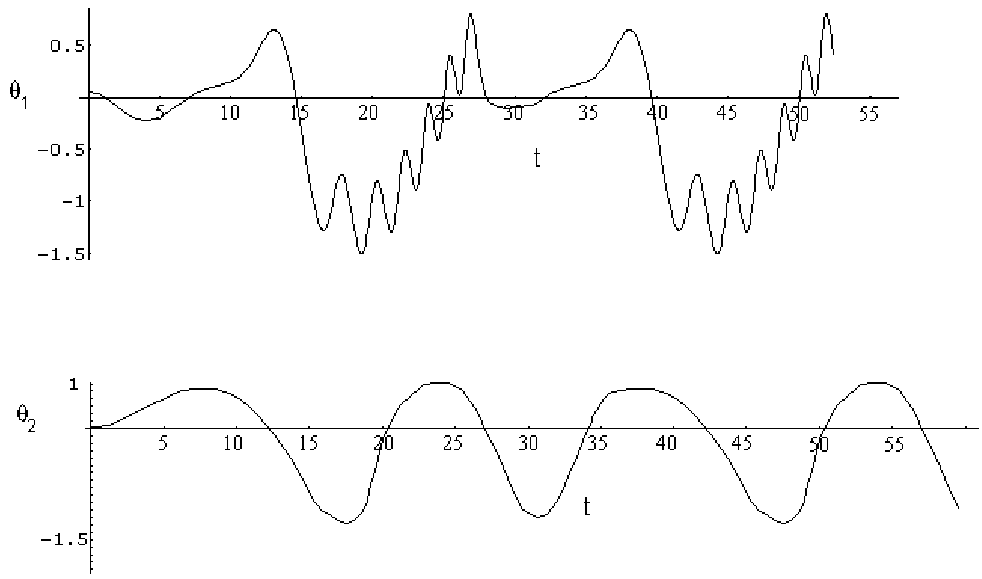

Fig. 3 Solutions θ_1(t) and θ_2 (t) for example 2.

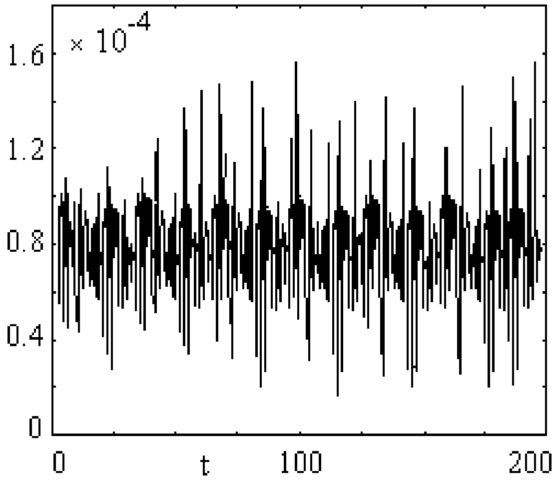

Fig. 4. Error function $e(t)$ between our solutions and Runge-Kutta solutions for Example 2.

Two examples are carried out for the simulation:

1. $m = 2$, $r = 2$, $s = 1$ ($\alpha = 0.3$, $\beta = 0.8$, $\gamma = 1.5$),
2. $m = 7$, $r = 1$, $s = 2$ ($\alpha = 0.038$, $\beta = 0.075$, $\gamma = 15$).

Figs. 2 and 3 represent the solutions $\theta_i(t)$, $i = 1, 2$ for considered examples. The solutions are stable, bounded and multi-periodic over the shown interval of time. For increasing t the shape of solutions remains unchangeable.

A comparison of the solutions with the numerical solutions obtained with the fourth-order Runge-Kutta scheme is performed. The numerical solutions are obtained by sewing the graphs on small intervals of time, with $\Delta t = 0.01$, to avoid deterioration of results. The error function $e(t)$ is defined as $e(t) = \sqrt{(\theta_1''(t) - \theta_1'(t))^2 + (\theta_2''(t) - \theta_2'(t))^2}$, where ($\theta_1'(t), \theta_2'(t)$) are our solutions and

$(\theta_1''(t),\theta_2''(t))$ the numerical solutions obtained by Runge-Kutta scheme, applied to the same set of equations. The error function is represented in Fig. 4 for Example 2.

The images of Fig 5 show the tendency of the pendulum to chaos for initial conditions in the interval [-3, 3].

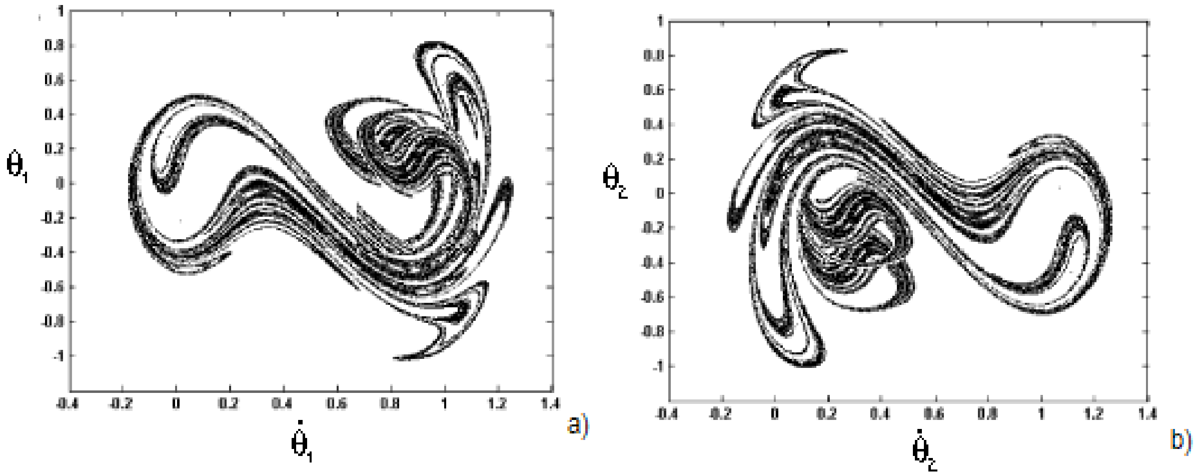

Fig. 5 Phase portraits $(\theta_1,\dot\theta_1)$ and $(\theta_2,\dot\theta_2)$ of the pendulum for initial conditions in the interval [-3, 3].

5. CONCLUSIONS

Theoretical and numerical results have been presented for determining the analytical representation of solutions of the nonlinear equations that govern the motion of a double pendulum.

The solutions are written as a linear superposition of Coulomb vibrations.

The solutions allow the possibility of investigating in detail the effects of changing the parameters values m, r, s. The solutions are stable, bounded and multi-periodic over the shown interval of time. The capability of the method can be extended to a qualitative analysis not only of the quasi-periodic behavior but also of the chaotic behavior of the pendulum. The results of this paper are encouraging to be applied also to other nonlinear dynamical systems with complex behavior [13, 14].

REFERENCES

1. BAKER, G.L., GOLLUB, J.P., *Chaotic dynamics. An introduction,* Cambridge Univ. Press, 1992.
2. SCHEIDI, R., TROGER, H., ZEMAN, K., *Coupled flutter and divergence bifurcation of a double pendulum,* Int. J. Non-Linear Mechanics, 19(2), 163-176, 1983.
3. DRAGANI, RACHELLE, *The Use of Pendulums in the Real World*, sciencing.com, https://sciencing.com/use-pendulums-real-world-8417546.
4. DONESCU, ST., *Coupled oscillators*, PhD Thesis, University of Bucharest, Mathematics Department, 2000.
5. MUNTEANU, L., BADEA, T., CHIROIU, V., *Linear equivalence method for the analysis of the double pendulum's motion*, Complexity International, vol.9, 2002.
6. TOMA, I., *Linear equivalence method and applications*, Ed. Flores, Bucharest, 1995.

7. TEODORESCU, P.P., TOMA, I., *An application of the linear equivalence method to the postcritical study of a Bernoulli-Euler bar*, Rev. Roum. Sci. Tech., 34(4), 309-321, 1989.
8. TEODORESCU, P.P., *Mechanical Systems. Classical Models*, vol. 1, Ed. Tehnică, Bucharest, 1984.
9. BOLOTIN, V.V., *Nonconservative problems of the theory of elastic stability*, Macmillan Comp. NY, pp.274-312, 1963.
10. COLLATZ, L., *The numerical treatment of differential equations*, Berlin, Springer Verlag, 1966.
11. MUNTEANU. L., DONESCU, St., *Introduction in the soliton theory. Applications in mechanics*, Publishing House of the Romanian Academy, 2002.
12. ABRAMOWITZ, M., STEGUN I.A., (eds.), *Handbook of mathematical functions*, U. S. Dept. of Commerce, 1984.
13. JENSEN, J.S., *Nonlinear dynamics of the follower-loaded double pendulum with added support-excitation*, Report no. 557, Technical Univ. of Denmark, 1997.
14. KAMEL, A.A., *Perturbation method in the theory of nonlinear oscillations*, Celestial Mechanics, vol.3, pp.90-106, 1970.

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