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ON THE DOUBLE PENDULUM FLEXIBLE MANIPULATOR

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Abstract By extending the linear equivalence method (LEM), the nonlinear equations of a two link flexible manipulator are obtained as a linear superposition of Coulomb vibrations. In addition, the nonlinear phenomena of dynamic responses such as internal resonance has investigated. We show that the solutions allow a qualitative analysis not only of the quasi-periodic behavior but also of the chaotic tendency of the flexible manipulator.

Key words: Flexible manipulator, double pendulum, Linear equivalence method (LEM), Coulomb vibrations, chaos.

1. INTRODUCTION

The double pendulum is an example of a time-independent system that exhibits chaotic behaviour. Different variants of the double pendulum include asymmetrical versions in which mass is displaced along the rod [1, 2]. In the last decade, the flexible manipulators of the double pendulum type have been the major concern for the researchers who tried to solve the vibration problem by improving the dynamic models and considering different loading conditions [3-5].

The choice of two link flexible manipulator as a model of double pendulum consists in the fact that 400 years after Galileo's initial work, these systems have again become an object of research as a chaotic system [6-9]. The double pendulum behaviour can be rich and complex, leading to the chaos. In this paper, the free and forced motion of the flexible manipulator is facilitated by using the linear equivalence method [10-13]. The LEM solutions of the non-simple-regular solutions of pendulum are describable as a linear superposition of Coulomb vibrations.

LEM is applicable for the solving of first-order differential nonlinear differential equations having algebraic nonlinearities and an arbitrary number of unknowns [14]. The integrable conservative systems can be analysed by using the Hamilton-Jacobi theory and the Morino formulation [15]. Smith and Morino [16] predict only simple harmonic limit-cycle solutions, but the Lie transformation method [17] extend the solutions for the cases of not-so-regular vibrations due to the fact that not only the zero-divisor terms but certain small-divisor terms can be included into analysis. The Lie transformation method is applicable to the Bolotin systems of equations, but is not adequate for the more complex systems as the equations of a double pendulum, because of the non-standard type of involved nonlinearities [18-24].

2. EQUATION OF MOTIONS

We consider a 2-beam manipulator similar to the system analysed in [21]. Fig. 1 shows a double pendulum type flexible manipulator. The beams have the lengths l_1 and l_2, the elastic moduli E_1 and E_2, second moment of area I_1 and I_2, cross sectional area A_1 and A_2. A beam is fixed at one end and attached to the motor on the other end, and the other beam is attached to the first motor from one end and to the second motor on the other. The motor and the payload mass at the end of the beams are considered as concentrated masses M_1 and M_2. The global system of coordinates is (X,Y) with (I,J) the unit vectors. A moving system of coordinates is attached to each beam $n=1,2$, and (i_n, j_n), $n=1,2$, are the unit vectors. The beams are modeled by the Euler–Bernoulli theory neglecting the effect of rotary inertia and shear deformation. The elastic deformation is denoted by $w_n(x,t)$, $n=1,2$. The relationships between the unit vectors of inertial and moving coordinate system for the both beams are

$$i_1 = \cos\theta_1 I + \sin\theta_1 J , \tag{1}$$

$$j_1 = -\sin\theta_1 I + \cos\theta_1 J , \tag{2}$$

$$i_2 = \cos(\theta_1 + w_1' + \theta_2) I + \sin(\theta_1 + w_1' + \theta_2) J , \tag{3}$$

$$j_2 = -\sin(\theta_1 + w_1' + \theta_2) I + \cos(\theta_1 + w_1' + \theta_2) J . \tag{4}$$

The end point (*R*) and the actual point *s* on the beams are

$$R_1 = l_1 i_1 + w_1' j_1 , \tag{5}$$

$$R_2 = R_1 + l_2 i_2 + w_2' j_2 , \tag{6}$$

$$s_1 = x_0 i_1 + (y_0 + w_1) j_1 , \tag{7}$$

$$s_2 = R_1 + x_0 i_2 + (y_0 + w_2) j_2 , \tag{8}$$

where (x_0, y_0) is the nondeformed position of an arbitrary point of the beam.

The potential energy of the flexible manipulator is composed of the elastic strain energy U_1, energy due to axial stretching U_2, potential energy of the link U_3 and potential energy of masses at the end of the link U_4, respectively

$$U_1 = \frac{1}{2}\sum_1^2 \int_0^{l_i} \mathrm{d}x E_i I_i w_i''^2 , \tag{9}$$

$$U_2 = \frac{1}{2}\sum_1^2 \int_0^{l_i} \mathrm{d}x E_i A_i w_i''^2 , \tag{10}$$

$$U_3 = \sum_1^2 \int_0^{l_i} \rho_i g \left(w_{(i-1)}' + x_0 \cos\alpha + (y_0 + w_i)\sin\alpha \right) \mathrm{d}x , \tag{11}$$

$$U_4 = M_i g \left(w_{(i-1)}' + l_i \cos\alpha \right) + w_i' \sin\alpha , \tag{12}$$

$$\alpha = \sum_1^2 (\theta_j + w_{(j-1)}') . \tag{13}$$

The kinetic energy of the flexible manipulator is

$$T = \frac{1}{2}\left(M_1 \dot{R}_1^T \dot{R}_1 + M_2 \dot{R}_2^T \dot{R}_2 + \int_0^{l_1} \rho_1 \dot{s}_1^T s_1 \mathrm{d}x + \int_0^{l_2} \rho_2 \dot{s}_2^T s_2)\mathrm{d}x \right). \tag{14}$$

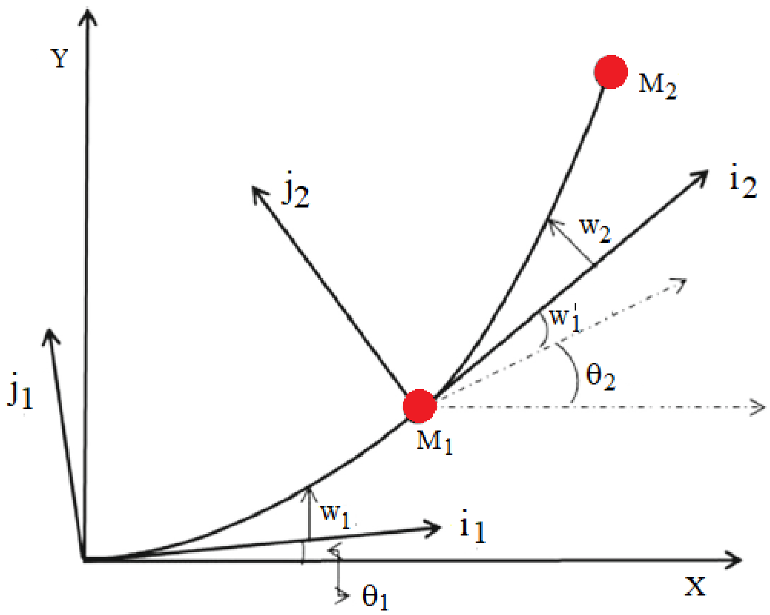

Fig. 1. Two-beam flexible manipulator.

The motion equations obtained from the Lagrange equations, are

$$\rho_1 A_1 (\ddot{w}_1 + x_0 \ddot{\theta}_1 - w_1 \dot{\theta}_1^2 + g) + E_1 I_1 w_1'''' - \frac{3}{2} E_1 A_1 w_1'' w_1'^2 = 0\,, \tag{15}$$

$$\int_0^{l_2} (\rho_2 A_2 \lambda_1 \mathrm{d}x + E_1 I_1 w' + M_2 \lambda_2) = 0\,, \tag{16}$$

$$(M_1 + M_2)\lambda_2 + \int_0^{l_2} \rho_2 A_2 \lambda_2 \mathrm{d}x - E_1 I_1 w_1'''' + E_2 I_2 w_2'''(0,t) + \frac{1}{2} E_1 I_1 w_1'^3(0,t) = 0\,, \tag{17}$$

$$\rho_2 A_2 \lambda_3 + E_2 I_2 w_2'''' - \frac{3}{2} E_2 A_2 w_2'^2 w_2'' = 0\,, \tag{18}$$

$$M_2 \lambda_2 - E_2 I_2 w_2 + \frac{1}{2} E_2 A_2 w_2'^3 = 0\,, \tag{19}$$

where

$$\lambda_1 = x_0 \ddot{w}_2 + x_0^2 \ddot{w}_1' + x_0^2 \ddot{\theta}_1 + x_0^2 \ddot{\theta}_2 + g a_1 + w_2 a_2 + w_2 a_3\,, \tag{20}$$

$$a_1 = x_0 \sin(\theta_2 + w_1')\,, \;\; a_2 = \cos(\theta_2 + w_1')\,, \tag{21}$$

$$a_3 = 2\dot{w}_2 \dot{\theta}_1 + 2\dot{w}_2 \dot{\theta}_2 + 2\dot{w}_2 \dot{w}_1' - w_2 \ddot{\theta}_1 + w_2 \ddot{\theta}_2 + w_2 \ddot{w}_1'\,, \tag{22}$$

$$\lambda_2 = \ddot{w}_1' + l_1 \ddot{\theta}_1 + g - w_1' \dot{\theta}_1^2\,, \tag{23}$$

$$\lambda_3 = \ddot{w}_1 + x_0 \ddot{w}_1' + x_0 \ddot{\theta}_1 + x_0 \ddot{\theta}_2 + a_1 - w_2 \dot{\theta}_1^2 - w_2 \dot{\theta}_2^2 - w_2 w_1'^2 - 2\dot{\theta}_1 \dot{\theta}_2 w_2 - 2\dot{\theta}_2 w_2 \dot{w}_2' + 2\dot{\theta}_1 w_2 \dot{w}_2'\,. \tag{24}$$

The initial and boundary attached conditions are

$$w_1(0,t)=0\,,\ \ w_1'(0,t)=0\,, \tag{25}$$

$$w_2(0,t)=0\,,\ \ w_2'(0,t)=0\,,\ \ E_2I_2w_2''(l_2,t)=0\,. \tag{26}$$

3. THE LINEAR EQUIVALENCE METHOD (LEM)

The system of equations (15-24) and (25, 26) is solved by the linear equivalence method (LEM). We begin with a brief explain of LEM in the spirit of [10].

To understand the method. consider a nonlinear Cauchy problem

$$\dot{z}=F\left(z\right),\quad z\left(t_0\right)=z_0,\quad t_0\in I\subset \mathrm{R}\,, \tag{27}$$

where $F_j\left(z\right)=\sum\limits_{|v|=1}^{\infty}a_{j\nu}z^{v}$, $a_{j\nu}\in I\subset R$ are n analytic functions. The $\nu=(\nu_1,\nu_2,...\nu_n)$ are multi-indexes of length n, and $z^{v}=\prod\limits_{j=1}^{n}z_j^{\nu_j}$.

A new variable $v(t,\sigma)$ is defined as an exponential transformation of real parameters σ

$$v\left(t,\sigma\right)=e^{<\sigma,z>}\,,\ \ <\sigma,z>=\sum_{j=1}^{n}\sigma_j z_j\,, \tag{28}$$

with $\sigma=[\sigma_1,\sigma_2,...\sigma_n]\in R^n$ and z_n , $n=1,2,...,6$.

The system (27) is transformed into by using (28)

$$\frac{\partial v}{\partial t}-\sum_{j=1}^{N_s}\sigma_j F_j(D)v=0\,, \tag{29}$$

with $F_j(t,D)$ a formal differential operator associated to $F_j(\sigma)$ given by

$$F_j\left(D\right)=\sum_{|\nu|=1}^{\infty}a_{j\nu}\frac{\partial^{|\nu|}}{\partial\sigma^{\nu}}\,,\ \ \frac{\partial^{|\nu|}}{\partial\sigma^{v}}=\frac{\partial^{v_1+v_2...+v_n}}{\partial\sigma_1^{v_1}\partial\sigma_2^{v_2}...\partial\sigma_n^{v_n}}\quad , \tag{30}$$

The linear equivalence transformation (28) is a solution of (27) and satisfies (29) and initial conditions

$$v\left(t_0,\sigma_1,\sigma_2,...\sigma_n\right)=\exp(\sigma_1 z_1^0+\sigma_2 z_2^0+...+\sigma_n z_n^0)\,. \tag{31}$$

The following results are reported [10]:

1. The transformation (28) where $z(t)$ is a solution of (27) satisfies (29) and the initial condition (31) $v\left(t_0,\sigma\right)=e^{<\sigma,z_0>}$.

2. If $\sup|a_{j\nu}(t)|<C$, the analytical solution $v\in C^1$, $t\in I$ of (29), (31), is determined using the known theories of partial linear equations. The solution $v(t,\sigma)$ has the form $v\left(t,\sigma\right)=e^{<\sigma,z>}$. The $z(t)$ satisfies (27) so the initial solution $z(t)$ is obtained from $v(t,\sigma)$.

This method is applied to (15-26). Let introduce the linear equivalence transformation (LEM) that depends on parameters $\sigma_i\in R$ $i=1,2,3,...,6$

$$v\left(t,\sigma\right)=\exp(\sigma_1 z_1+\sigma_2 z_2+\sigma_3 z_3+...+\sigma_6 z_6)\,,\ \ \sigma_i\in R\,, \tag{32}$$

By inserting (32) into (15-26) we have

$$\frac{\partial v}{\partial t}=\sum_{n=1}^{6}(\sum_{p=1}^{6}\sigma_n a_{np}\frac{\partial v}{\partial \sigma_p}+\sum_{p,q=1}^{6}\sigma_n b_{npq}\frac{\partial^2 v}{\partial \sigma_p \partial \sigma_q}+\sum_{p,q,r=1}^{6}\sigma_n c_{npqr}\frac{\partial^3 v}{\partial \sigma_p \partial \sigma_q \partial \sigma_r}+ \\ +\sum_{p,q,r,l=1}^{6}\sigma_n d_{npqrl}\frac{\partial^4 v}{\partial \sigma_p \partial \sigma_q \partial \sigma_r \partial \sigma_l}+\sum_{p,q,r,l,m=1}^{6}\sigma_n e_{npqrlm}\frac{\partial^5 v}{\partial \sigma_p \partial \sigma_q \partial \sigma_r \partial \sigma_l \partial \sigma_m}), \tag{33}$$

with initial conditions

$$v(0,\sigma)=\exp(\sigma_1 z_1^0+\sigma_2 z_2^0+...+\sigma_6 z_6^0). \tag{34}$$

The $v(t,\sigma)$ has the form

$$v(t,\sigma)=1+\sum_{k=1}^{6}\sum_{i=1}A_k^i\frac{\sigma_k^i}{i!}+\sum_{\substack{k,l=1\\k\neq l}}^{6}\sum_{i,j=1}A_k^i A_l^j\frac{\sigma_k^i\sigma_l^j}{i!\,j!}+ \\ +\sum_{\substack{k,l,m=1\\k\neq l\neq m}}^{6}\sum_{i,j,r=1}A_k^i A_l^j A_r^r\frac{\sigma_k^i\sigma_l^j\sigma_m^r}{i!\,j!\,r!}+\sum_{\substack{k,l,m,n=1\\k\neq l\neq m\neq n}}^{6}\sum_{i,j,r,s=1}A_k^i A_l^j A_r^r A_n^s\frac{\sigma_k^i\sigma_l^j\sigma_m^r\sigma_n^s}{i!\,j!\,r!\,s!}. \tag{35}$$

Inserting (35) into (33) and taking use of (34), the $A_n(t)$, $n=1,2,3,...,6$ are determined by equating the terms of the same power in σ and t. So, we will have

$$A_n(t)=\sum_{k,\eta=0}\{(\mu t)^{k+1}\tilde{A}_{nk}(\eta)\Phi_k(\mu t,\eta)+(\mu t)^k\tilde{B}_{nk}(\eta)\psi_k(\mu t,\eta)\}, \tag{36}$$

for $k=0,1,2,3,...k_{\max}$, $\eta=0,1,2,3,...,6.$

The $k_{\max}$ depend of η and is relevant for capturing the contribution of all nonlinearities of governing equations. So, we consider $k_{\max}=334$. For $k_{\max}>334$ the computing complicates without bringing significant terms in solutions. In (36), $\Phi_k(\mu t,\eta)$ and $\psi_k(\mu t,\eta)$ are

$$\Phi_k(\mu t,\eta)=\sum_{m=k+1}(\mu t)^{m-k-1}A_m^{(k)}(\eta),$$

$$\psi_k(\mu t,\eta)=\sum_{m=k+1}(\mu t)^{m-k-1}B_m^{(k)}(\eta), \tag{37}$$

and $\mu=\mu(\eta)$

$$\mu=\sum_{j=1}^{6}\alpha_j\lambda_j\;,\;\sum_{l=1}^{6}\alpha_j=\eta+1\;,\;\alpha_j\geq 0,\; j=1,2,3,...,6\;. \tag{38}$$

In (38) we have $\lambda_1=p_1$, $\lambda_2=-p_1$, $\lambda_3=p_2$, $\lambda_4=-p_2$, $\lambda_5=\omega$, $\lambda_6=-\omega$, where $\pm ip_1$, $\pm ip_2$ $\pm i\omega$ are the roots of the characteristic equation.

The unknowns $\tilde{A}_{nk}$, $\tilde{B}_{nk}$ in (3.10) are found as

$$\tilde{A}_{nk}(\eta)=C_k(\eta)B_{nk}(\eta),\;\tilde{B}_{nk}(\eta)=C_k(\eta)C_{nk}(\eta), \tag{39}$$

where $C_k(\eta)$ verify the recurrence relation

$$C_k(\eta) = \frac{\sqrt{k^2+\eta^2}}{k(2k+1)} C_{k-1}(\eta), \tag{40}$$

$$C_0(\eta) = \frac{p\,|\Gamma(1+i\eta)|}{2!}, \tag{41}$$

with Γ the gamma function and

$$|\frac{\Gamma(1+i\eta)}{\Gamma(2)}|^2 = \prod_{n=0}[1+\frac{\eta^2}{(2+n)^2}]^{-1}. \tag{42}$$

The constants $A_m^{(k)}$, $B_m^{(k)}$ are related by

$$B_m^{(k)}(\eta) = m\Delta A_m^{(k)}(\eta), \tag{43}$$

where

$$A_{k+1}^{(k)} = 1, \quad A_{k+2}^{(k)} = \frac{\eta}{k+1}, \quad (m+k)(m-k-1)A_m^{(k)} = 2\eta A_{m-1}^{(k)} - A_{m-2}^{(k)}, \quad m > k+2 \ , \tag{44}$$

The constants $B_{nk}(\eta)$, $C_{nk}(\eta)$ depend on initial conditions and on coefficients from (43). From (42) and $(37)_2$ we obtain

$$\psi_k(\mu t,\eta) = \sum_{m=k+1} (\mu t)^{m-k-1} B_m^{(k)}(\eta). \tag{45}$$

The function $\psi(\mu t,\eta)$ is linked to the $\Phi(\mu t,\eta)$ by

$$\Psi(\mu t,\eta) = \mu\frac{d}{dt}\Phi(\mu t,\eta) = \mu\dot{\Phi}(\mu t,\eta) \ . \tag{46}$$

By noting

$$F_k(\mu t,\eta) = C_k(\eta)\ (\mu t)^{k+1}\Phi_k(\mu t,\eta), \tag{47}$$

(36) becomes

$$A_n(t) = \sum_{\eta,k=0}\{B_{nk}(\eta)F_k(\mu t,\eta) + C_{nk}(\eta)\dot{F}_k\ (\mu t,\eta)\}]. \tag{48}$$

We obtain

$$F_0(\mu t,0) = \sin\mu t, \quad \dot{F}_0(\mu t,0) = \cos\mu t \ . \tag{49}$$

The LEM representation of the solutions of equations (15-24) and conditions (25, 26) is

$$z_n(t) = A_n(t) = \sum_{k,\eta=0}\{B_{nk}(\eta)F_k(\mu t,\eta) + C_{nk}(\eta)\dot{F}_k\ (\mu t,\eta)\}, \tag{50}$$

where $F_k(\mu t,\eta)$ have the form of Coulomb wave functions. Due to this fact, we call the functions $F_k(\mu t,\eta)$, the Coulomb vibrations. As mentioned above, the solutions of nonlinear equations that govern the motion of a double pendulum are written as a linear superposition of Coulomb vibrations.

4. NUMERICAL INVESTIGATIONS AND CONCLUSIONS

We assess the efficiency of the LEM analysis in computing the LEM representations of solutions for flexible robot. Fig. 2 represents the modal displacements $w_1(t)$ and $w_2(t)$ for the beam 1 and beam 2, vibration modes 1 and 2. Displacement frequencies are higher for the beam 2. We observe that the displacement amplitudes decrease according to the mode rank. A 15 Hz frequency is characteristic of all modal responses. A 1000 Hz frequency is reached for the third mode of beam 2.

The solutions are stable, bounded and multi-periodic over the shown interval of time. For increasing t the shape of solutions remains unchangeable.

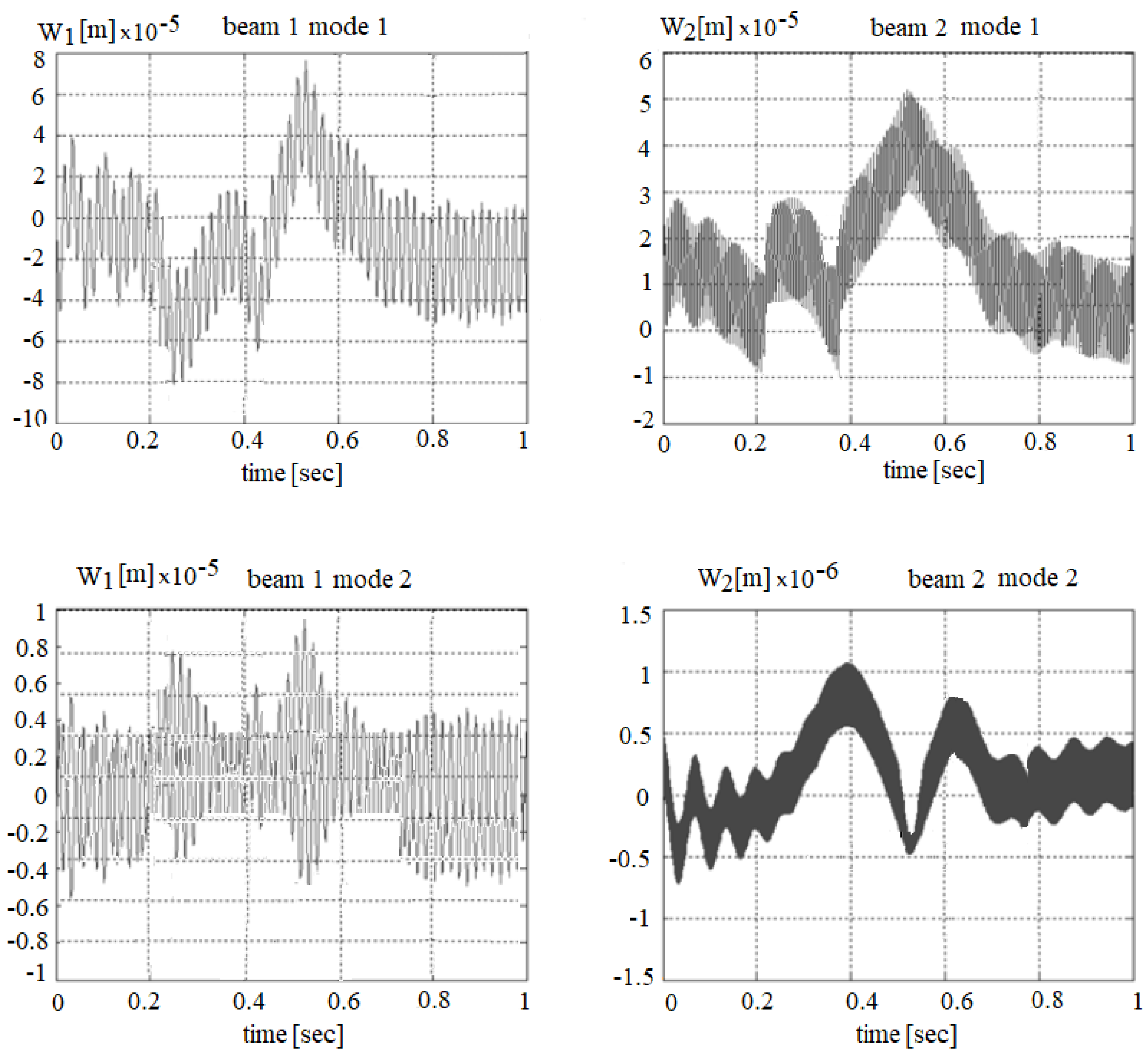

Fig.2 The LEM solutions w_1 and w_2.

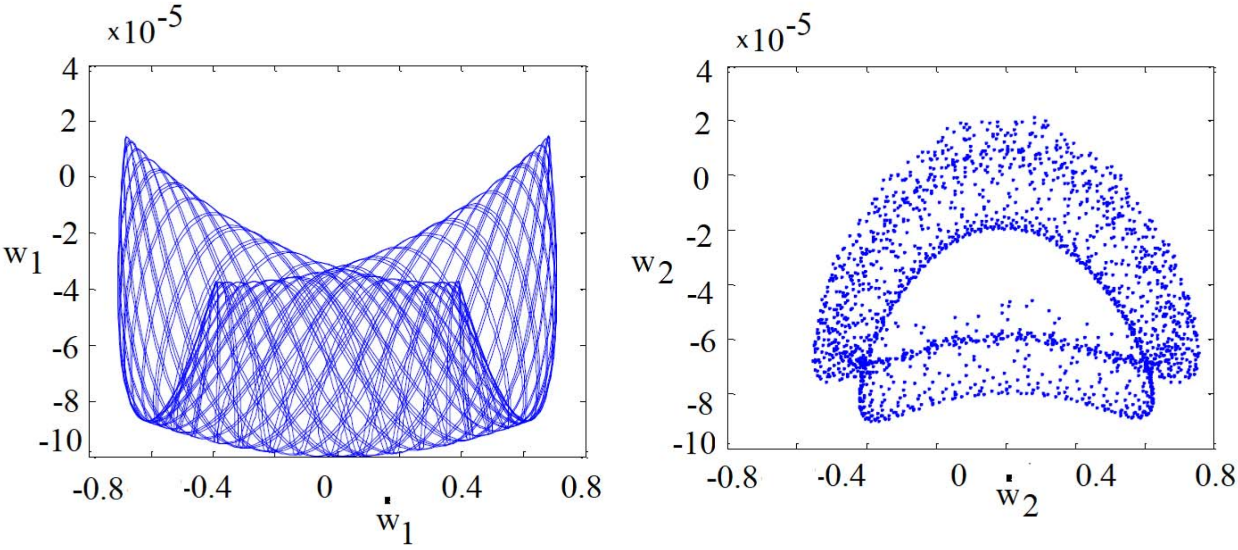

Fig.2. Poincare sections $(w_1, \dot{w}_1)$ and $(w_2, \dot{w}_2)$.

The flexible manipulator has different behaviors, as a function of initial conditions. We should conclude that the flexible manipulator can tend to chaotic motions for certain initial conditions. Use of the Poincare mappings help us to decide if an orbit is ordered or chaotic.

The 2-D Poincare sections $(w_1, \dot{w}_1)$ and $(w_2, \dot{w}_2)$ are represented in Fig.2 for a set of initial conditions (25) of the form

$$w_1(0,t) = 0.8\,,\;\; w_1'(0,t) = -0.6\,. \tag{25}$$

5.CONCLUSIONS

Theoretical and numerical results have been presented for determining the analytical solutions of the nonlinear equations that govern the motion of a double pendulum flexible manipulator by using the linear equivalence method (LEM). The solutions are written as a linear superposition of Coulomb vibrations. The general expression of the solutions can describe not only the stable behavior of the pendulum, but also the unstable behavior for same cases with chaotic behavior. So, the capability of LEM can be extended to the analysis of the chaotic behavior of the pendulum.

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