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CA18232 – MATHEMATICAL MODELS FOR INTERACTING DYNAMIC NETWORKS - Part 1 Cnoidal Theory

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Abstract The intention of this work is to discuss some mathematical and computational models used to integrate and interpret heterogeneous engineering data, understanding fundamental principles of dynamics in the frame of the action CA18232 – Mathematical models for interacting dynamic networks. Thus, such tools are now routinely used in the theoretical and experimental systematic investigation of dynamical systems.

Key words: CA18232, dynamics networks.

1. INTRODUCTION

Many physical, biological, chemical, financial or even social phenomena can be described by dynamical systems. It is quite common that the dynamics arises as a compound effect of the interaction between sub-systems in which case we speak about coupled systems. This Action shall study such interactions in particular cases from three points of view: the abstract approach to the theory behind these systems, applications of the abstract theory to coupled structures like networks, neighbouring domains divided by permeable membranes, possibly non-homogeneous simplicial complexes, etc., modelling real-life situations within this framework.

The purpose of this Action is to bring together leading groups in Europe working on a range of issues connected with modelling and analysing the mathematical models for dynamical systems on networks. It aims to develop a semigroup approach to various (non-)linear dynamical systems on networks as well as numerical methods based on modern variational methods and applying them to road traffic, biological systems, and further real-life models. The Action also explores the possibility of estimating solutions and long-time behaviour of these systems by collecting basic combinatorial information about underlying networks.

The intention of this work is to discuss some mathematical and computational models used to integrate and interpret heterogeneous engineering data, understanding fundamental principles of dynamics. Thus, such tools are now routinely used in the theoretical and experimental systematic investigation of dynamical systems.

The main research direction is based on the study of the nonlinear dynamical systems with application to mechanical engineering as the mechanics, the acoustics, biomechanics and nanomechanics. So, there are discussed selected topics from the dynamics of sonic composites and the role of defects on the full band-gaps generation, a 3D spherical acoustic cloaking and a new class of sonic composites based on the auxetic materials. The Sommerfeld effect is analysed for dynamics of a building subjected to vibrations, and the chaos-hyperchaos transition in the behaviour of this building which completes the analysis. Applications to biomechanics refer to the motion of blood in small vessels and the anisotropy of Young's modulus in human bones. Applications to nanomechanics refer to the size effects in the buckling of the carbon nanotubes and the analysis of the behaviour of the reinforcement carbon nanotube beams. We add the modeling of viscoplastic flow in some technological processes. In this first part, the cnoidal theory - the principal tool of exploring the interacting dynamic networks is presented.

2. THE CNOIDAL METHOD

Dynamic network analysis is an emergent scientific field that brings together problems related to the field of nonlinear dynamical systems. These problems request strong knowledge related to applied mathematics and mechanics, since vibrations of real engineering systems are dynamic phenomena described by differential or partial differential equations which are nonlinear and often strongly nonlinear.

There are two aspects of this field. The first is the solving the nonlinear dynamical equations and the dynamic analysis of data. The second is the utilization of simulation to address issues of network dynamics. An early study of the dynamics of link utilization in very large-scale complex networks provides evidence of dynamic centrality, dynamic motifs, stability and chaos in structural interactions.

One method used to solve the nonlinear motion equations is the cnoidal method. The cnoidal approach is coupled to certain nonlinear theories of deformable media and the genetic algorithm in order to solve the practical problems selected from the field of mechanical engineering [1-3].

The inverse scattering theory generally solves certain nonlinear differential equations, which have cnoidal solutions [4]. The mathematical and physical structure of the inverse scattering transform solutions has been extensively studied in both one and two dimensions [5- 10]. The theta-function representation of the solutions is expressed as a linear superposition of Jacobi elliptic functions and additional terms which include interactions among them.

The method is reducible to a generalization of the Fourier series with the cnoidal functions as the fundamental basis function. The cnoidal functions are much richer than the trigonometric or hyperbolic functions, that is, the modulus m of the cnoidal function, $0 \leq m \leq 1$, can be varied to obtain a sine or cosine function $(m \cong 0)$, a Stokes function $(m \cong 0.5)$ or a solitonic function, sech or tanh $(m \cong 1)$. In the cnoidal method, the arc length of the ellipse is related to the Jacobi elliptic integrals of the first and the second kinds, respectively.

$$E(z) = \int_0^z \frac{\sqrt{(1-k^2x^2)}\mathrm{d}x}{\sqrt{(1-x^2)}},\ F(z) = \int_0^z \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

for $0 < k < 1$. Jacobi compares the integral

$$v = \int_0^{\varphi} \frac{\mathrm{d}\theta}{(1-m\sin^2\theta)^{1/2}}, \tag{1}$$

with $0 \leq m \leq 1$, to the integral

$$w = \int_0^{\varphi} \frac{\mathrm{d}t}{(1-t^2)^{1/2}}, \tag{2}$$

and observed that (2) defines the inverse of the trigonometric function sin (if we use the notations $t=\sin\theta$ and $\psi=\sin w$). The Jacobi elliptic functions are usually written $\mathrm{sn}(v,m)$ and $\mathrm{cn}(v,m)$, where $0\le m\le 1$. The angle φ is called the amplitude $\varphi=\mathrm{am}\,u$

$$\mathrm{sn}\,v=\sin\varphi,\ \ \mathrm{cn}\,v=\cos\varphi. \tag{3}$$

For $m=0$, we have

$$v=\varphi,\ \ \mathrm{cn}(v,0)=\cos\varphi=\cos v,$$

$$v=\varphi\ \mathrm{sn}(v,0)=\sin\varphi=\sin v,\ \ \mathrm{dn}(v,0)=1, \tag{4}$$

and for $m=1$

$$v=\mathrm{arcsech}(\cos\varphi),\ \ \mathrm{cn}(v,1)=\mathrm{sech}v,$$

$$\mathrm{sn}(v,1)=\tanh v,\ \ \mathrm{dn}(v,1)=\mathrm{sech}v. \tag{5}$$

The $\mathrm{sn}\,v$ and $\mathrm{cn}\,v$ are periodic functions with the period

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{(1-m\sin^2\theta)^{1/2}}=4\int_0^{\pi/2}\frac{\mathrm{d}\theta}{(1-m\sin^2\theta)^{1/2}},$$

where later integral is the complete elliptic integral of the first kind

$$K(m)=\int_0^{\pi/2}\frac{\mathrm{d}\theta}{(1-m\sin^2\theta)^{1/2}}. \tag{6}$$

The period of the function $\mathrm{dn}\,v$ is $2K$. For $m=0$ we have $K(0)=\pi/2$. For increasing of m, $K(m)$ increases monotonically

$$K(m)\approx\frac{1}{2}\log\frac{16}{1-m}.$$

Now we consider the function $\wp(t)$ introduced by Weierstrass in 1850. This function verifies the equation

$$\dot{\wp}^2=4\wp^3-g_2\wp-g_3, \tag{8}$$

where the point means differentiation with respect to t.

We noted with e_1,e_2,e_3 the real roots of the equation $4y^3-g_2y-g_3=0$ with $e_1>e_2>e_3$. Then, (8) becomes

$$\dot{\wp}^2=4(\wp-e_1)(\wp-e_2)(\wp-e_3), \tag{9}$$

$$g_2=2(e_1^2+e_2^2+e_3^2),\quad g_3=4e_1e_2e_3,\ \ e_1+e_2+e_3=0.$$

When

$$\Delta=g_2^3-27g_3^2>0, \tag{10}$$

then (9) admits the elliptic Weierstrass function as a particular solution

$$\wp(t+\delta';g_2,g_3)=e_2-(e_2-e_3)\mathrm{cn}^2(\sqrt{e_1-e_3}t+\delta'), \tag{11}$$

where δ' is an arbitrary real constant. The equation (11) is reducing to the Jacobi elliptic function

If we impose initial conditions to (9)

$$\wp(0) = \theta_0, \quad \wp'(0) = \theta_{p0}, \tag{12}$$

then a linear superposition of (11) is also a solution

$$\theta_{lin} = 2\sum_{k=0}^{n} \alpha_k \mathrm{cn}^2[\omega_k t; m_k], \tag{13}$$

where the angular frequencies ω_k, and amplitudes α_k depend on θ_0, θ_{p0} .

When $\Delta < 0$ the solution of (9) becomes

$$\wp = e_2 + H_2 \frac{1+\mathrm{cn}(2t\sqrt{H_2}+\delta')}{1-\mathrm{cn}(2t\sqrt{H_2}+\delta')},$$

$$m = \frac{1}{2} - \frac{3e_2}{4H_2}, \quad H_2 = 3e_2^2 - \frac{g_2}{4}.$$

When $\Delta = 0$, it results $e_1 = e_2 = c$, $e_3 = -2c$, and the solution of (9) becomes

$$\wp = c + \frac{3c}{\sinh^2(\sqrt{3c}t + \delta')}. \tag{14}$$

Consider the Weierstrass equation with a polynomial of n degree in $\theta(t)$

$$\dot{\theta}^2 = P_n(\theta). \tag{15}$$

For a biquadratic polynomial, $n = 4$, four real zeros are obtained: two real and two purely imaginary zeros, four purely imaginary zeros or four genuinely complex zeros. For $n = 5$ the solutions depends also on the zeros of the polynomial. For all cases the solutions are expressed in terms of Jacobi elliptic functions, the hyperbolic and trigonometric functions [1, 5].

The general solution of (15) is written in the terms of the theta function representation [11]

$$\theta(x,t) = \frac{2}{\lambda} \frac{\mathrm{d}^2}{\mathrm{dx}^2} \log \Theta_n(\eta_1, \eta_2, ..., \eta_n), \tag{16}$$

where $\lambda = \alpha / 6\beta$, and Θ is *the theta function* defined as

$$\Theta_n(\eta_1, \eta_2, ..., \eta_n) = \sum_{M \in (-\infty,\infty)} \exp\left(\mathrm{i} \sum_{i=1}^{n} M_i \eta_i + \frac{1}{2} \sum_{i,j=1}^{n} M_i B_{ij} M_j \right), \tag{17}$$

with $1 \le j \le n$ and

$$\eta_j = k_j x - \omega_j t + \phi_j, \ 1 \le j \le n, \tag{18}$$

where k_j are the wave numbers, ω_j are the frequencies and ϕ_j are the phases. The vectors of wave numbers, frequencies and constant phases are given by

$$k = [k_1, k_2, ..., k_n], \quad \omega = [\omega_1, \omega_2, ..., \omega_n], \quad \phi = [\phi_1, \phi_2 ..., \phi_n], \quad \eta = [\eta_1, \eta_2, ..., \eta_n], \tag{19}$$

$$\eta = kx - \omega t + \phi. \tag{20}$$

$$M\eta = Kx - \Omega t + \Phi, \ M = [M_1, M_2, ..., M_n], \ K = Mk, \ \Omega = M\omega, \Phi = M\phi.$$

The integer components in M are the integer indices in (17). The matrix B can be decomposed in a diagonal matrix D and an off-diagonal matrix O, that is

$$B = D + O\,. \tag{21}$$

THEOREM [1] *The solution* $\theta(x,t)$ *of* (*15*) *is written as*

$$\theta(x,t) = \frac{2}{\lambda}\frac{\partial^2}{\partial x^2}\log\Theta_n(\eta) = \theta_{lin}(\eta) + \theta_{int}(\eta)\,, \tag{22}$$

where θ_{lin} *represents a linear superposition of cnoidal functions*

$$\theta_{lin}(\eta) = \frac{2}{\lambda}\frac{\partial^2}{\partial x^2}\log G(\eta)\,, \tag{23}$$

$$G(\eta) = \sum_{M}\exp\left(\mathrm{i}M\eta + \frac{1}{2}M^T DM\right), \tag{24}$$

θ_{int} *represents a nonlinear interaction among the cnoidal functions*

$$\theta_{\mathrm{int}}(\eta) = 2\frac{\partial^2}{\partial t^2}\log\left(1 + \frac{F\ \ (\eta, C)}{G\ \ (\eta)}\right), \tag{25}$$

$$F\ \ (\eta, C) = \sum_{M^\alpha} C\exp\left(\mathrm{i}M\eta + \frac{1}{2}M^T DM\right), \tag{26}$$

$$C = \exp\left(\frac{1}{2}M^T OM\right) - 1\,. \tag{27}$$

Consider now a nonlinear system of equations

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = F_i(\theta_1, \theta_2, ..., \theta_n), \quad i = 1, ..., n, \quad n \geq 3\,, \tag{29}$$

with $x \in \mathrm{R}^n$, $t \in [0,T]$, $T \in \mathrm{R}$. The function F is given by

$$\begin{aligned} F_i = \sum_{p=1}^{n} a_{ip}\theta_p + \sum_{p,q=1}^{n} b_{ipq}\theta_p\theta_q + \sum_{p,q,r=1}^{n} c_{ipqr}\theta_p\theta_q\theta_r + \\ + \sum_{p,q,r,l=1}^{n} d_{ipqrl}\theta_p\theta_q\theta_r\theta_l + \sum_{p,q,r,l,m=1}^{n} e_{ipqrlm}\theta_p\theta_q\theta_r\theta_l\theta_m + \ldots\,, \end{aligned} \tag{30}$$

$i = 1, 2, ..., n$, $a, b, c...$ constants.

The system of equations (29) can be reduced to Weierstrass equations of the type (15). The transformation

$$\theta = 2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log\Theta_n(t)\,, \tag{31}$$

applied to $\Theta_n(t)$

$$\Theta_1 = 1 + \exp(\mathrm{i}\omega_1 t + B_{11})\,,$$

$$\Theta_2 = 1 + \exp(\mathrm{i}\omega_1 t + B_{11}) + \exp(\mathrm{i}\omega_2 t + B_{22}) + \exp(\omega_1 + \omega_2 + B_{12})\,,$$

$$\begin{aligned} \Theta_3 = 1 + \exp(\mathrm{i}\omega_1 t + B_{11}) + \exp(\mathrm{i}\omega_2 t + B_{22}) + \exp(\mathrm{i}\omega_3 t + B_{33}) + \exp(\omega_1 + \omega_2 + B_{12}) + \\ + \exp(\omega_1 + \omega_3 + B_{13}) + \exp(\omega_2 + \omega_3 + B_{23}) + \exp(\omega_1 + \omega_2 + \omega_3 + B_{12} + B_{13} + B_{23}), \end{aligned} \tag{32}$$

and

$$\Theta_n = \sum_{M\in(-\infty,\infty)} \exp(\mathrm{i}\sum_{i=1}^{n} M_i\omega_i t + \frac{1}{2}\sum_{i<j}^{n} B_{ij} M_i M_j), \tag{33}$$

$$\exp B_{ij} = \left(\frac{\omega_i - \omega_j}{\omega_i + \omega_j}\right)^2, \ \exp B_{ii} = \omega_i^2. \tag{34}$$

Let us write the solution (31) as

$$\theta(t) = 2\frac{\partial^2}{\partial t^2}\log\Theta_n(\eta) = \theta_{lin}(\eta) + \theta_{int}(\eta), \tag{35}$$

with $\eta = -\omega t + \phi$. The θ_{lin} represents a linear superposition of cnoidal waves

$$\theta_{lin} = \sum_{l=1}^{m} \alpha_l \mathrm{cn}^2[\eta; m_l], \tag{36}$$

with

$$q = \exp\left(-\pi\frac{K'}{K}\right), \ K = K(m) + \int_0^{\pi/2}\frac{\mathrm{d}u}{\sqrt{1\text{-}m\sin^2 u}}, \ K'(m_1) = K(m), \ \ m + m_1 = 1. \tag{37}$$

The term θ_{int} represents a nonlinear superposition or interaction among cnoidal waves

$$2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log\left(1 + \frac{F(t)}{G(t)}\right) \approx \frac{\beta_k \mathrm{cn}^2(\eta, m_k)}{1 + \gamma_k \mathrm{cn}^2(\eta, m_k)}. \tag{38}$$

or

$$\theta_{int}(x,t) = \frac{\sum_{k=0}^{m}\beta_k \mathrm{cn}^2[\eta; m_k]}{1 + \sum_{k=0}^{n}\lambda_k \mathrm{cn}^2[\eta; m_k]}\ . \tag{39}$$

As a result, the cnoidal method yields to solutions consisting of a linear superposition (36) and a nonlinear superposition (39) of cnoidal functions. Details of the soliton equations can be found [12-27].

3. CONCLUSIONS

This paper introduces the fundamental ideas underlying some mathematical methods in the frame of the action CA18232 – Mathematical models for interacting dynamic networks.

The solitons are localized waves that conserve their properties even after interaction among them, and then act somewhat like particles. The equations which describe the solitons have interesting properties: an infinite number of local conserved quantities, an infinite number of exact solutions expressed in terms of the Jacobi elliptic functions or the hyperbolic functions and the simple formulae for nonlinear superposition of explicit solutions. Such equations were considered integrable or more accurately, exactly solvable. Substantial parts of this chapter are based on the papers [11-27].

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REFERENCES

1. ABLOWITZ, M. J., RAMANI, A., SEGUR, H., *A connection between nonlinear evolution equations and ordinary differential equations of P-type I, II*, J. Math. Phys., 21, 715–721, 1006–1015, 1980.
2. ABLOWITZ, M. J., SEGUR, H., *Solitons and the inverse scattering transforms*, SIAM, Philadelphia, 1981.
3. ABLOWITZ, M. J., CLARKSON, P. A., *Solitons, nonlinear evolution equations and inverse scattering*, Cambridge Univ. Press, Cambridge, 1991.
4. MUNTEANU, L., DONESCU, St., *Introduction to Soliton Theory: Applications to Mechanics*, Book Series Fundamental Theories of Physics, vol.143, Kluwer Academic Publishers, Dordrecht, Boston (Springer Netherlands) 2004.
5. COURANT, R., HILBERT, D., *Methods of Mathematical Physics, II*, Interscience, New York, 1962.
6. CRĂŞMĂREANU, M., *Cylindrical Titeica curves imply forced harmonic oscillators*, Balkan J. of Geometry and its Applications, 7, 37–42, 2002.
7. DRAZIN, G., JOHNSON, R. S., *Solitons. An introduction*, Cambridge Univ. Press, 1989.
8. DRAZIN, P. G., Solitons, Cambridge Univ. Press, 1983.
9. DUNCAN, D. B., EIBECK, J. C., FEDDERSEN, H., WATTIS, J.A.D., *Solitons on lattices*, Physica D, 68, 1-11, 1993.
10. JASON H.V. NGUYEN, PAUL DYKE, DE LUO, BORIS A. MALOMED, RANDALL G. HULET, *Collisions of matter-wave solitons*, Nature Physics 2012.
11. DI CARLI, A., D. C. COLQUHOUN, HENDERSON, G., FLANNIGAN, S., OPPO, G-L., DALEY, A.J., KUHR, ST. HALLER, E., *Excitation Modes of Bright Matter-Wave Solitons*, Phys. Rev. Lett. 123, 123602 - September 2019.
12. DAUXOIS, M.P.T., *Physics of Solitons*, Cambridge University Press, Cambridge, England, 2006.
13. SERKIN, V., T. BELYAEVA, T., *Excitation Modes of Bright Matter-Wave Soliton*, Optik (Stuttgart) 176, 38, 2019.
14. GAMAYUN, O., SEMENYAKIN, M., *Soliton splitting in quenched classical integrable systems*, J. Phys. A 49, 335201, 2016.
15. OPANCHUK, K.L., REID, M.D., DRUMMOND, P.D., Phys. Rev. Lett. 122, 203604, 2019.
16. LIU, I.K., CHUAN G.S., TAKEUCHI, H., *Phase diagram of solitons in the polar phase of a spin-1 Bose-Einstein condensate*, Phys. Rev. Research 2, 033506- September 2020.
17. BARKMAN, M., SAMOILENKA, A., WINYARD, T., BABAEV, E., *Ring solitons and soliton sacks in imbalanced fermionic systems*, Phys. Rev. Research 2, 043282- November 2020.
18. LEVKOV, D.G., MASLOV, V.E., NUGAEV, E.Y., *Chaotic solitons in driven sine-Gordon model*, 2020.
19. CALOGERO, A., DEGASPERIS, F., *Spectral Transform and Solitons*, North-Holland, Amsterdam, 1982.
20. ABLOWITZ, M.J., KAUP, D.J., NEWELL, A.C., SEGUR, H., *Construction of solutions of the sine-Gordon equation by means of Fredholm determinants*, Stud. Appl. Math. 53, 249, 1974.
21. GARDNER, C.S., GREENE, J.M., KRUSKAL, M.D., MIURA, R.M., *Method for Solving the Korteweg-de Vries Equation*, Phys. Rev. Lett. 19, 1095, 1967.
22. CHENG-AN CHENCHEN-LUNG HUNG, *Observation of modulational instability and the soliton formation in two-dimensional Bose gases*, 2019.

23. KROLIKOWSKI, W., LUTHER-DAVIES C.B., TSCHUDI, D., *Annihilation of photorefractive solitons*, Optics Letters 1998.
24. ISLAM, M.N., SOCCOLICH, C.E., *Billiard-ball soliton interaction gates*, Opt. Lett. 16,1991.
25. BRADLEY, C. SACKETT, R. *Bose-Einstein Condensation of Lithium: Observation of Limited Condensate Number*, Physical Review Letter1997.
26. JASON H.V. NGUYEN, PAUL DYKE, DE LUO, BORIS A. MALOMED, RANDALL G. HULET, Collisions of matter-wave solitons, Nature Physics, 10, 918-922, 2014.
27. W. KRÓLIKOWSKIB, B. LUTHER-DAVIES, C. DENZ, T. TSCHUDI, *Annihilation of photorefractive soliton*s, Opt. Lett. 23, 1998.

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