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ON THE BOUNCING BALL DYNAMICS

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Abstract In this paper, we analyze the dynamics of two bouncing balls (a light ball and a much heavier ball) vertically aligned and simultaneously dropped on a moving lower boundary. The balls are subjected to gravitational field. The floor-collisions together to the velocity discontinuity to impacts and uncertainties arising from the instant of generation of the next impact, can lead the motion of the system ball to chaos.

Key words: Vibro-impact dynamics, bouncing ball, chaos.

1. INTRODUCTION

The vibro-impact motion when the bodies undergo a collision has applications in robotics, multi-body dynamics [1-5], the transport of granular matter [6], the railway bogies dynamics [7] and other problems. The impact can provoke discontinuities in the motion of the system being characterized by short durations, high forces and rapid dissipation of energy with large accelerations and decelerations.

The contact, as opposite to impact, is a continuous process over a finite time and describes the situation where the bodies come in touch with each other [8].

Harter [9] experimentally investigates the motion of two balls by modeling each ball as a nonlinear neo-Hookeean spring, with the nonlinear coefficients obtained by experimental data. Patrício [10] investigates a Hertzian contact between two spheres and between the lower sphere and the ground, and Cross [11] considered a linear stiff-spring model to explain the experimental results for the motion of a basketball and a tennis ball.

The problem of two and three body impact is extended to multi-body chain collision [12]. Boechler *et al.* [13] modelled by Korteweg–de Vries-like models the 1D diatomic granular crystal composed of compressed elastic beads as a chain of nonlinear oscillators.

In the present work, we study the vibro-impact problem of two bouncing balls moving in the gravitational field in a finite region with a unilateral constraint done by a moving floor. In [14-17] the displacement of the floor is taken as a piecewise linear periodic function of time, and in [18] the displacement of the floor is a quadratic or a cubic function of time.

In this article, the displacement of the floor is considered to be $y(t) = \sin(t)$.

Although the problem seems simple, the chain collision impact for multi-body systems, still has plenty of unsolved issues.

2. THE COLISSION MODEL

The motion of a single ball between two impacts is [8, 15, 19]

$$m\ddot{z} = -mg\,, \quad y = y(t)\,, \tag{1}$$

$$z(\tau_i) = y(\tau_i)\,,$$

$$\dot{z}(\tau_i^+) - \dot{y}(\tau_i) = -R\left(\dot{z}(\tau_i^-) - \dot{y}(\tau_i)\right), \tag{2}$$

where $y(t)$ is the displacement of the floor, τ_i is the time of the i impact, and + and - note the left and right-sides of the floor, $0 \le R < 1$ is the restitution coefficient and $g = 9.81\text{ms}^{-1}$. Duration of the impact is neglected.

Consider now, two balls of masses m_1 and m_2, $m_1 << m_2$ and radii r_1 and r_2, $r_1 << r_2$ respectively, vertically aligned with initial starting heights z_1 and z_2 above the moving floor (Fig. 1).

The motion of the floor is given by $y(t) = \sin(t)$.

The lower ball impacts the floor at time $t_0 = \sqrt{2(z_2 - r_2)/g}$ with velocity $v_0 = -\sqrt{2g(z_2 - r_2)}$. This ball rebounds with velocity

$$u_0 = -e_2 v_0 = e_2\sqrt{2g(z_2 - r_2)} > 0\,, \tag{3}$$

where e_2 is the coefficient of restitution between the lower ball and the flour, and $v_0 < 0$ is the velocity of the upper ball. The balls are separated by the distance Δz. The second impact appears at time

$$t_1 = t_0 - \frac{\Delta z}{(1+e_2)v_0} = t_0 + \Delta t\,. \tag{4}$$

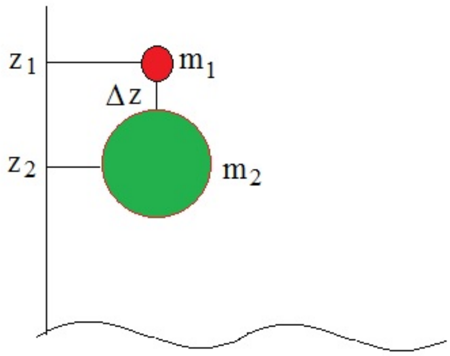

Fig. 1. Two bouncing balls.

The velocity of the lower ball and the upper ball prior to impact are $u_2 = u_0 - g\Delta t$ and $u_1 = -gt_1$, respectively. After impact the velocities of balls are v_1 and v_2, respectively. These velocities are determined from the conservation of momentum

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2\,, \tag{5}$$

and the restitution coefficient between the balls

$$e_{12} = \frac{v_2 - v_1}{u_2 - u_1}. \tag{6}$$

In the case of elastic collision $e_2 = e_{12} = 1$. We get

$$v_1 = \frac{\mu u_1 + u_2 + e_{12}(u_2 - u_1)}{\mu + 1}, \quad v_2 = \frac{\mu u_1 + u_2 + \mu e_{12}(u_1 - u_2)}{\mu + 1}, \tag{7}$$

with $\mu = \frac{m_1}{m_2}$. For $\Delta z \to 0$, we obtain

$$v_1 \to \frac{\mu - e_2 - e_{12}(1+e_2)}{\mu + 1} v_0, \quad v_2 \to= \frac{\mu - e_2 + \mu e_{12}(1+e_2)}{\mu + 1} v_0. \tag{8}$$

For $\mu \to 0$, we have

$$v_1 \to -((1+e_2)e_{12} + e_2)v_0, \quad v_2 \to -e_2 v_0. \tag{9}$$

Relation (9) predict the post-impact velocity of the upper ball in the limit $\mu \to 0$. The dynamic analysis uses as control parameters the restitution coefficients e_2, e_{12} and the masses of the balls ration μ.

3. RESULTS

The simulation results are obtained for the separation distance $\Delta \in [0,70]$mm and $z_2 \in [0.2,1]$m, $\mu \in [0.1, 0.4]$, $e_2 \in [0.5, 0.9]$ and $e_{12} \in [0.5, 0.9]$. The velocity discontinuities to impacts can lead the motion of the bouncing balls to chaos. To show this, the rebound velocity is plotted with respect to v_0 for both balls in Fig. 2.

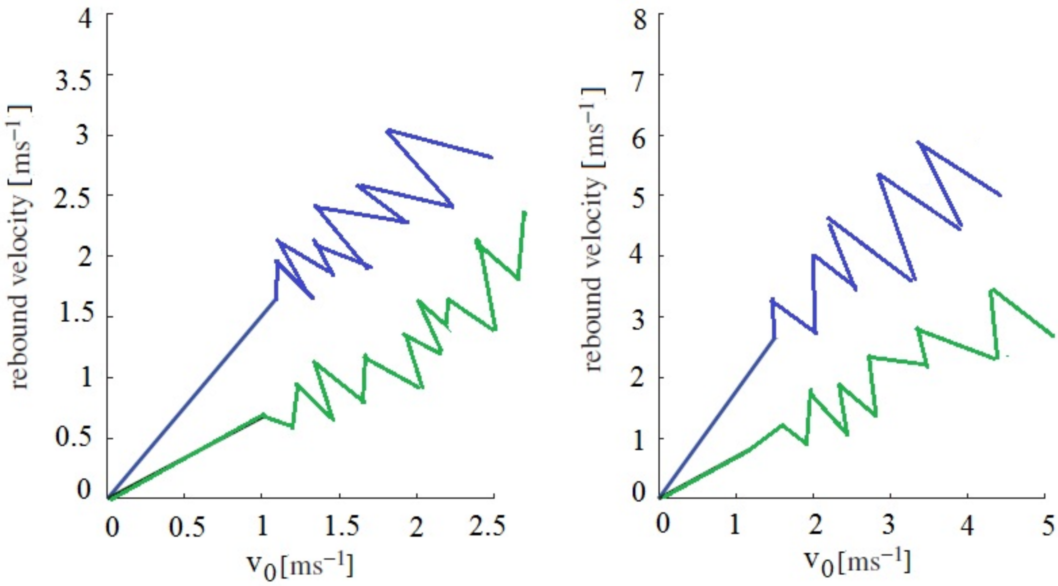

Fig. 2. The variation of the rebound velocity with respect to v_0 for both balls for $\Delta z = 0$.

We observe, that for a given initial separation Δz ,we see that the ratio of the post-impact to pre-impact velocity of each ball is independent of the magnitude of the pre-impact velocity. This result agrees to observation of Berdeni in [20].

Next, we compute the ratio of post impact velocities to the pre-impact velocity of both balls $\frac{v_1}{|v_0|}$ and $\frac{v_2}{|v_0|}$, respectively, is displayed in Fig. 3 as a function of Δz for $z_2 = 1\text{m}$. The velocity ratio for both ball exhibit irregularities for $\Delta z < 0.07$ for the first ball, and $\Delta z < 0.06$ for the second one.

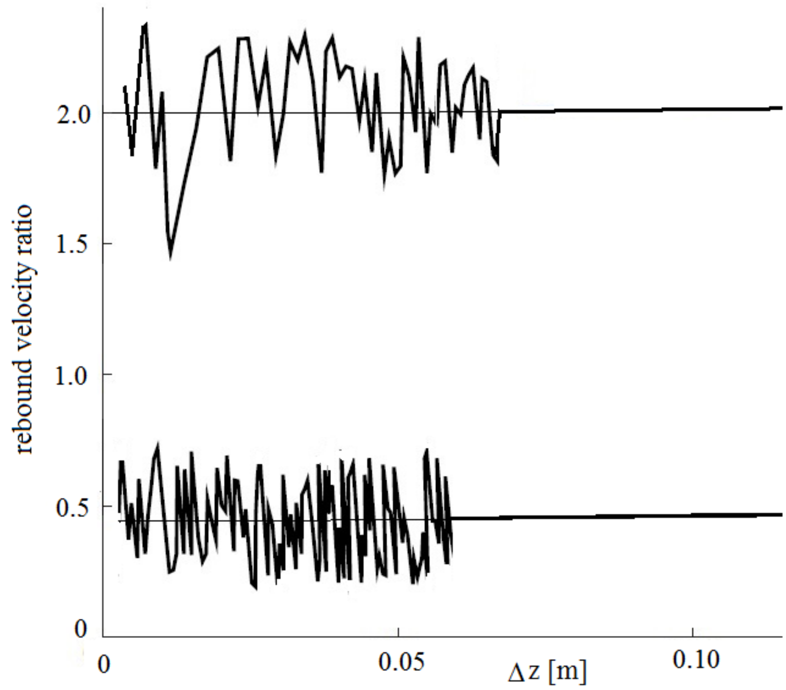

Fig. 3. Rebound velocities of the balls expressed as the rebound velocity ratio.

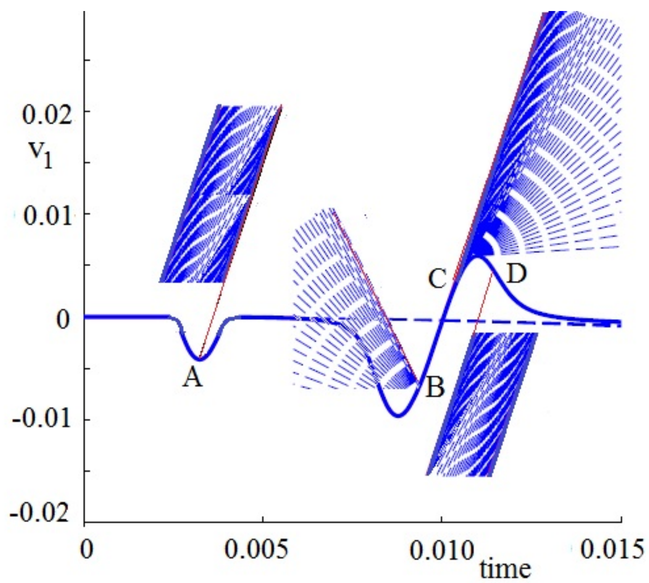

Fig. 4. Bifurcation diagram corresponding to points A, B, C and D for v_1 .

The bifurcation diagram for v_1 corresponding to four arbitrary points A,B,,C and D is displays in Fig.4 for the restitution coefficients $e_2 = 0.78$, $e_{12} = 0.63$ and $\mu \in (0.31, 0.39)$ after ten impacts.

The points A,B,C and D correspond to four critical values of μ, namely $\mu_{cr} = 0.327$, $\gamma_{cr} = 0.328$, $\gamma_{cr} = 0.329$ and $\gamma_{cr} = 0.331$, respectively.

In Fig. 5, the plot of velocity v_1 versus Δz corresponding to the bifurcation diagrams in points A,B,C and D from Fig. 3, are presented for 20s.

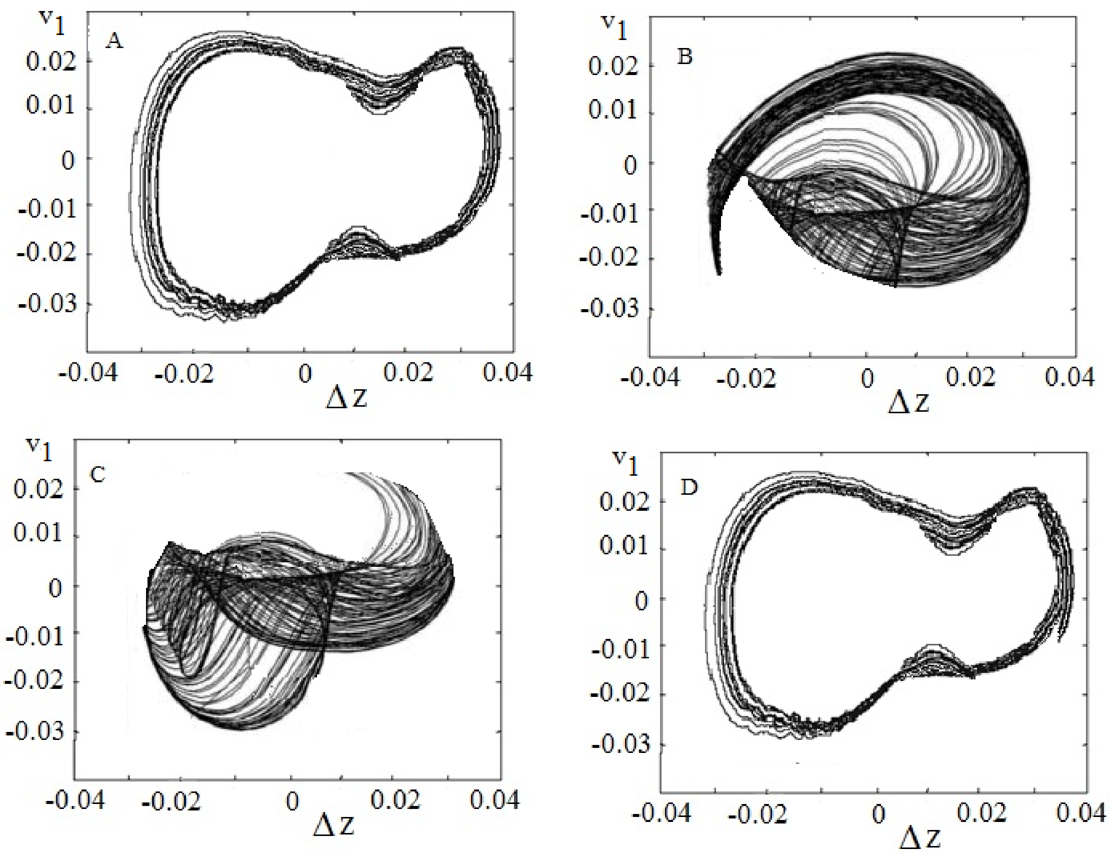

Fig. 5. The phase portrait $(v_1, \Delta z)$ for 20s.

4. CONCLUSIONS

In this paper, the motion of two bouncing balls, i.e. a light ball and a much heavier ball, simultaneously dropped on a moving lower boundary, is analyzed. The system is sensitive to the impact discontinuity, leading it to chaos. The task of the paper is the understanding of irregular temporal dynamics of the bouncing balls seen on the long run.

The tendency to chaos is supported by noisy motions of the balls, so that the deterministic behavior of the system is blurred by a random behavior.

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