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ON THE BURGERS' EQUATION WITH APPLICATION TO WAVE PROPAGATION IN SOFT TISSUES

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Abstract. The propagation of sound waves in soft biological tissues (blood, veins, kidney, liver, lung, etc.) is described in this paper by using the Burgers' equation. The propagation depends on the properties of the tissue at the ultrasonic range of frequencies over 20 kHz, with emphasis on the range 1–10 MHz. The propagation of waves in soft tissues is used for diagnostic and tissue therapy. Utility of the Burgers' equation to sonification technique is highlighted next to a medical image used to surgical operation.

Key words: Burgers' equation; Biological tissues: Sonification.

1. INTRODUCTION

Burgers equation belongs to a class of nonlinear partial differential equations which possess the solitons or solitary waves as elementary solutions, like Korteweg–de Vries (KdV) and Schrödinger equations. These localized solutions conserve their properties after interaction with other waves, and are expressed by the Jacobi elliptic functions (cnoidal solutions) or the hyperbolic functions (solitons) with simple formulae for superposition [1].

The Burgers' equation describes the propagation of sound waves of finite amplitudes in solids, liquids and gases being characterized by distortion of the localized pressure field as the wave travels [2, 3]. The equation was first introduced by Harry Bateman in 1915 and studied by Johannes Martinus Burgers in 1948 [4, 5, 6]. The application of the Burgers' equation to nonlinear acoustics is shown by Cole in 1949 [7] and Hopf in 1950 [8]. The Burgers' equation describes the effect of dissipative effects to the finite-amplitude waves in acoustics, thermodynamics and hydrodynamics [9-14].

The Burgers' equation is written as

$$w_t = w_{xx} + ww_x , \tag{1}$$

where w is the acoustic velocity, and the subscript means the differentiation with specified variable.

This equation reminds of the Riemann equation

$$p_t + pp_x = 0 , \tag{2}$$

as a particular case of (1), where p is the pressure deviation of a medium (air, for example). The Riemann equation is used in nonlinear acoustic waves propagation for which the viscosity of the medium is not taken into account.

The solutions of (1) can be obtained by using the Hopf - Cole transformation

$$w(x,t)=\frac{2}{u}u_x\,, \tag{3}$$

where $u(x,t)$ verifies the equation $u_{,t}=u_{,xx}$. The common initial conditions attached to (1) are

$$w=f(x) \quad \text{at} \quad t=0\,,\ -\infty<x<\infty\,. \tag{4}$$

The general form of solutions of (1) is [15, 16]

$$w(x,t)=2\frac{\partial}{\partial x}\ln F(x,t)\,, \tag{5}$$

$$F(x,t)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\exp\left[-\frac{(x-\xi)^2}{4t}-\frac{1}{2}\int_0^{\xi}f(\xi')d\xi'\right]\mathrm{d}\xi\,. \tag{6}$$

From (5) and (6) the following solutions of (1) are obtained

$$w(x,t)=\lambda+\frac{2}{x+\lambda t+A}\,, \tag{7}$$

$$w(x,t)=\frac{4x+2A}{x^2+Ax+2t+B}\,, \tag{8}$$

$$w(x,t)=\frac{6(x^2+2t+A)}{x^3+6xt+3Ax+B}\,, \tag{9}$$

$$w(x,t)=-\lambda+A\frac{\exp\left[A(x-\lambda t)\right]-B}{\exp\left[A(x-\lambda t)\right]+B}\,, \tag{10}$$

$$w(x,t)=-\gamma+2A\tanh\left[A(x-\lambda t)+B\right]. \tag{11}$$

$$w(x,t)=-\frac{\lambda}{\lambda^2 tA}\left[2\tanh\frac{\lambda x+B}{\lambda^2 t+A}-\lambda x-B\right], \tag{12}$$

with A,B,λ, the arbitrary constants.

Certain generalization of (1) are [17]:

- Burgers-Huxley equation

$$u_t+\alpha u u_x=\mu u_{xx}+\mu u+\eta u^2-\delta u^3\,, \tag{13}$$

- Kolmogorov-Petrovsky-Piskunov equation (Fisher equation)

$$u_t=\mu u_{xx}+\mu u+\eta u^2\,, \tag{14}$$

- Korteweg-de Vries-Burgers equation

$$u_t+\alpha u u_x+u_{xxx}=\mu u_{xx}\,, \tag{15}$$

- Kuramoto-Sivashinsky equation

$$u_t+u u_x+\alpha u_{xx}+\beta u_{xxx}+\lambda u_{xxxx}=0\,, \tag{16}$$

The Burgers' equation can take the form of the modified Westervelt equation

$$w_x-\frac{\beta}{c_0^2}ww_{\tau}=\frac{b}{2\rho_0 c_0^3}ww_{\tau\tau}\,, \tag{17}$$

with $\tau = t - x / c_0$ the retarded time, c_0 the velocity of sound propagation in the linear approximation, β the Burgers coefficient which quantifies the nonlinear effects, ρ_0 density of medium and b the coefficient of shear viscosity. The equation (17) is used to describe the propagation of waves in soft tissues [18-20].

The exact solutions of nonlinear equations (13-17) are obtained by different methods such as tanh method [21], pseudospectral method [22], inverse scattering method [23], cnoidal method [1], Bakland transformation [24] and variational methods [25]. Also, there are some numerical iterative methods that are converging rapidly.

2. BURGERS' EQUATION

The propagation of waves in soft tissues is described by Burgers' equation (17)

$$w_x - \frac{\beta}{c_0^2} w w_\tau - \frac{b}{2\rho_0 c_0^3} w w_{\tau\tau} = 0 . \tag{18}$$

This equation admits semi-analytical solutions almost for any initial condition $w(0, x) = w_0$. These solutions provide a good opportunity to evaluate the properties of the wave motion through soft tissues. Equation (18) admits localized solutions known as cnoidal waves. These solutions conserve their properties at interaction with other waves. Like other equations (Schrödinger, Korteweg–de Vries, etc.) equation (18) has an infinite number of local conserved quantities, an infinite number of exact solutions and the simple formulae for nonlinear superposition of explicit solutions. The solution of (18) is searched under the form

$$w = \sum_{j=1}^{l} \alpha_j \mathrm{cn}^j(m_j, \eta_i) + \frac{\sum_{j=1}^{l} \gamma_j \mathrm{cn}^j(m_j, \eta_i)}{1 + \sum_{j=1}^{l} \lambda_j \mathrm{cn}^j(m_j, \eta_i)} , \tag{19}$$

where $\eta = kx - \omega t + \tilde{\phi}$, l is a finite number of degree of freedom of the cnoidal functions, $0 \le m \le 1$ is the moduli of the Jacobean elliptic function, ω is frequency and $\tilde{\phi}$ the phase, k the wave number.

For the tissue, the most important part of the acoustic attenuation is achieved through the viscous mechanism. In this case, the attenuation is given by the classical frequency free absorption coefficient α_a / f^2, where f is frequency [26]

$$\frac{\alpha_a}{f^2} = \frac{b}{2\rho_0 c_0^3} . \tag{20}$$

So, the equation (18) can be rewritten in the form

$$w_x - \frac{\beta}{c_0^2} w w_\tau - \frac{\alpha_a}{f^2} w w_{\tau\tau} = 0 . \tag{21}$$

The total attenuation in the tissue combines the absorption and scattering losses

$$\alpha = \alpha_a + \alpha_s , \tag{22}$$

where α_a is the amplitude absorption coefficient, and α_s is the amplitude scattering coefficient.

The intensity attenuation coefficient is given by

$$\mu = \mu_a + \mu_s = 2\alpha . \quad (23)$$

3. SOLUTIONS

Usually, the amplitude absorption coefficient in tissues are expressed as a function of frequency

$$\alpha_a = af^n , \quad (24)$$

where a, n are constants and f is frequency. For liver and brain for examples, the following expressions are used [26]

$$\alpha_{a\,liver} = 3.3 + 0.2 f^3 , \quad (25)$$

$$\alpha_{abrain} = 0.23 + 0.006 f^4 . \quad (26)$$

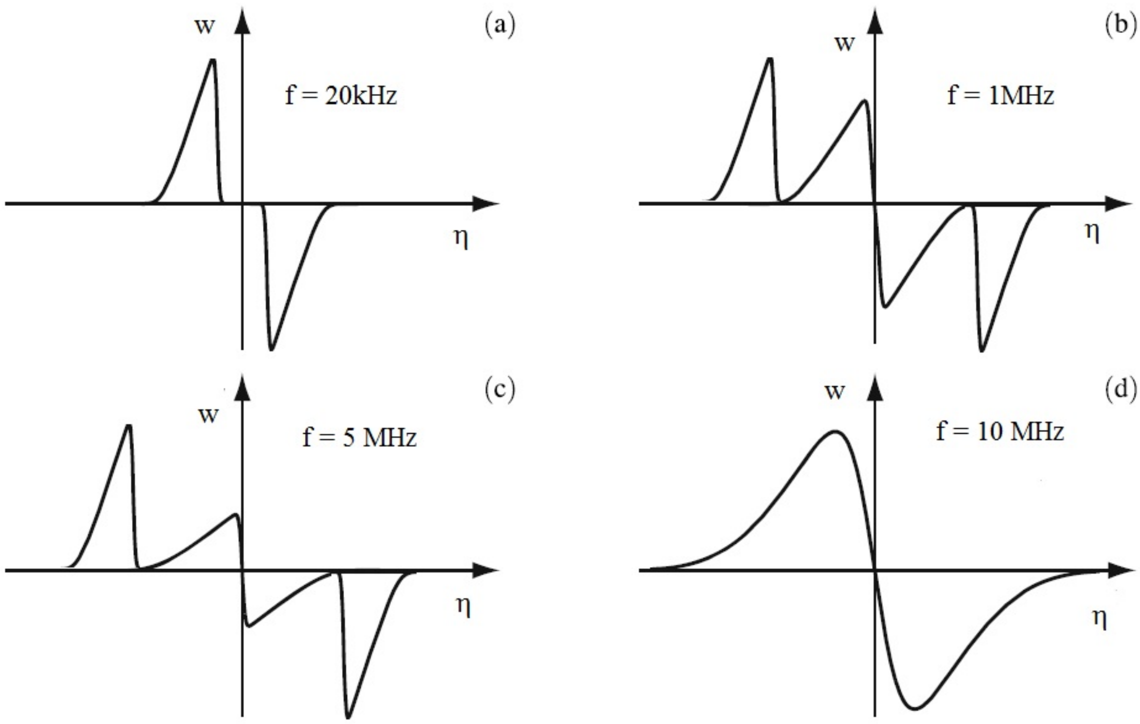

Fig. 1. The solutions (27) of the Burgers' equation for different frequencies for (25).

In the following we stop to $l = 2$ in (19), and we will see that there are no sensible improvements in solutions for $l > 2$.

$$w = \alpha_1 \mathrm{cn}(m_1, \eta_1) + \alpha_2 \mathrm{cn}^2(m_2, \eta_2) + \frac{\gamma_1 \mathrm{cn}(m_1, \eta_1) + \gamma_2 \mathrm{cn}^2(m_2, \eta_2)}{1 + \lambda_1 \mathrm{cn}(m_1, \eta_1) + \lambda_2 \mathrm{cn}^2(m_2, \eta_2)} , \quad (27)$$

where the parameters α, γ, λ are expressed in terms of β, c_0, ρ_0, b and w_0 .

The solutions (27) of the Burgers' equation for different frequencies are displayed in Fig.1, for (25), and Fig. 2 presents the solutions for different frequencies for (26), respectively. The horizontal axes is $\eta = kx - \omega t + \tilde{\phi}$, and the vertical scale is the acoustic velocity.

The Burgers' equation (18) or (21) can be reduced to Riemann equation (2) if the viscosity vanishes ($b = 0$). In this case, the solution (27) is reduced to

$$p(x,t) = \frac{1}{t}(x - y_p(x,t)), \tag{28}$$

where $y_p(x,t)$ is the coordinate of the absolute minimum of the function

$$f(y,x,t) = s_0(y)t + \frac{1}{2}(y-x)^2, \tag{29}$$

with $s_0(x)$ the initial potential of the pressure field.

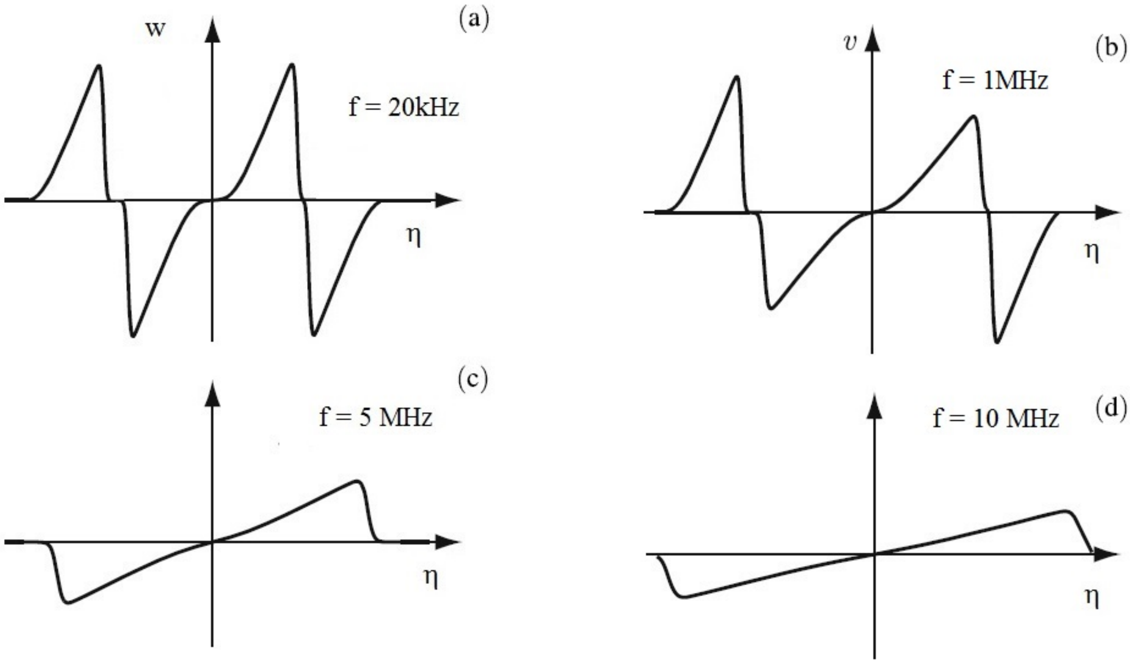

Fig. 2. The solutions (26) of the Burgers' equation for different frequencies for (26).

4. PROPERTIES OF THE BURGERS' EQUATION

The properties of the Burgers' equation print a special behavior to the wave propagation. The space and time translations symmetry, the odd reflection symmetry and the Galilean invariance are the main symmetries of the equation [27].

If $w(x,t)$ verifies the equation (18) then $w(x+a,t)$ and $w(x,t+\tau)$, with a,τ arbitrary space and time translations, respectively, also verify the equation.

It results that (18) verifies the translational symmetry

$$w(x,t) \Leftrightarrow w(x+a,t+\tau). \tag{30}$$

If $w(x,t)$ verifies (18) then $-w(-x,t)$ also verify the equation (the odd reflection symmetry)

$$w(x,t) \Leftrightarrow -w(-x,t). \tag{31}$$

The Galilean invariance means that the Burgers' equation is invariant to the transformation

$$w(x,t) = V + \tilde{w}(x - Vt, t)), \quad (32)$$

where $\tilde{w}(x',t)$ is the velocity of the particles in the coordinate system x' moving with the velocity V. The invariance (32) allows to derive a family of simultaneously solutions starting from a given solution. For a known solution with the initial condition $\tilde{w}_0(x)$, we also known new solutions with the initial conditions

$$w_0 = \tilde{w}_0(x) + V. \quad (33)$$

Two families of simultaneously solutions starting from a given solution are displayed in Fig. 3 and Fig. 4, for 0.1 MHz and 0.5 MHz, respectively.

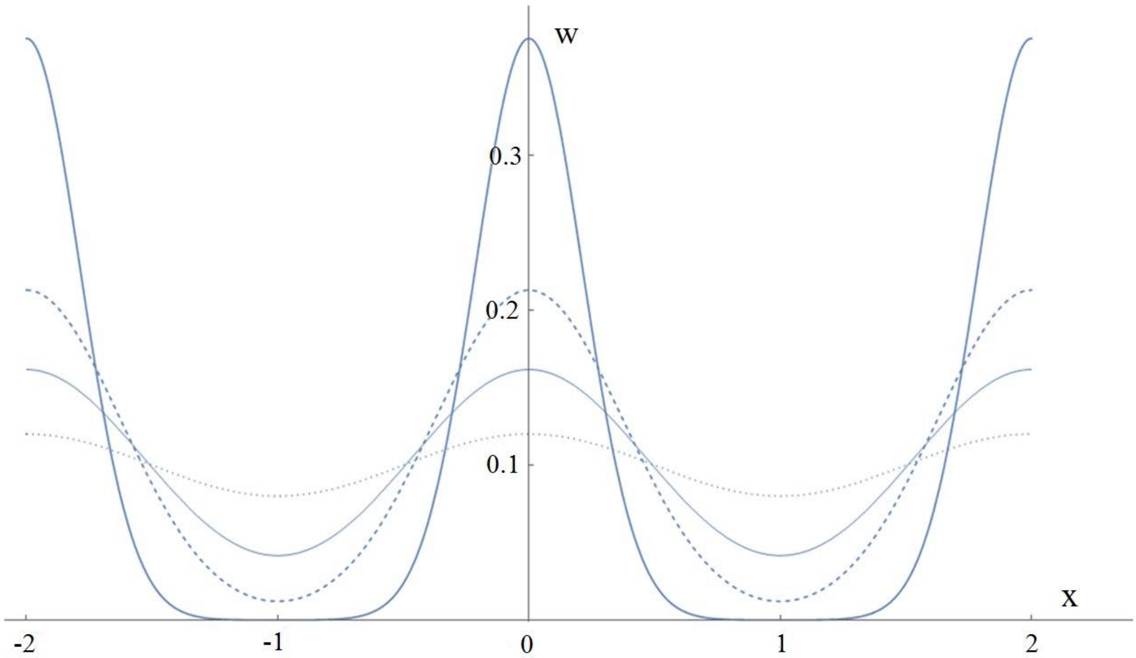

Fig. 3. A family of solutions starting from a given solution of the Burgers' equation at 0.1 MHz.

The symmetry properties are useful to evaluate the variation of the Burgers' equation under a change of the length scale and the magnitudes of the initial field.

If the initial velocity $w_0(x)$ has the characteristic length scale l, and the magnitude U

$$w_0(x) = U u_0\left(\frac{x}{l}\right), \quad (34)$$

then after the transformation

$$s = \frac{x}{l}, \;\; u(s,t) = \frac{w(sl,t)}{U}, \quad (35)$$

the equation (18) takes the form

$$\frac{U}{l}u_s - \frac{\beta}{c_0^2}uu_\tau - \frac{b}{2\rho_0 c_0^3}uu_{\tau\tau} = 0, \;\; u(s,0) = u_0(s). \quad (36)$$

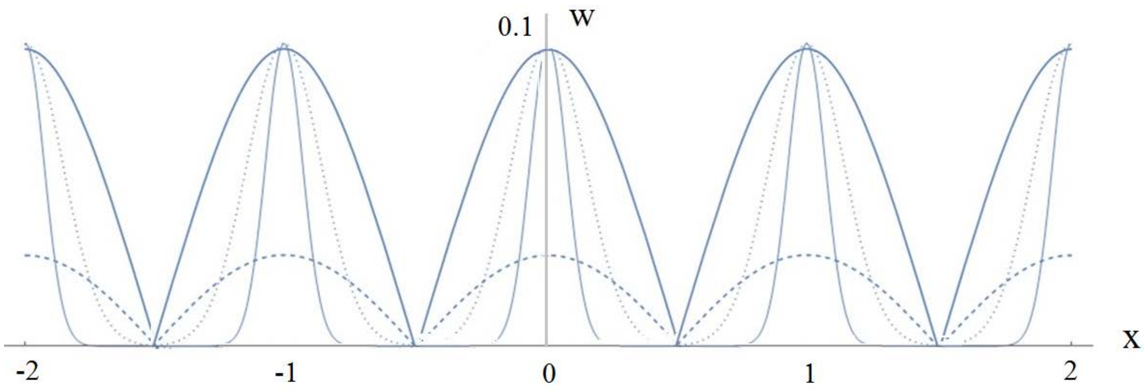

Fig. 4. A family of solutions starting from a given solution of the Burgers' equation at 0.5 MHz.

An interesting property of the Burger' equation is the creation and annihilation of the wave picks due to a change of frequency $\omega_0 \pm \omega_1$. If the shift is positive the pick is created, and if the shift is negative the pick is annihilated (Fig. 5).

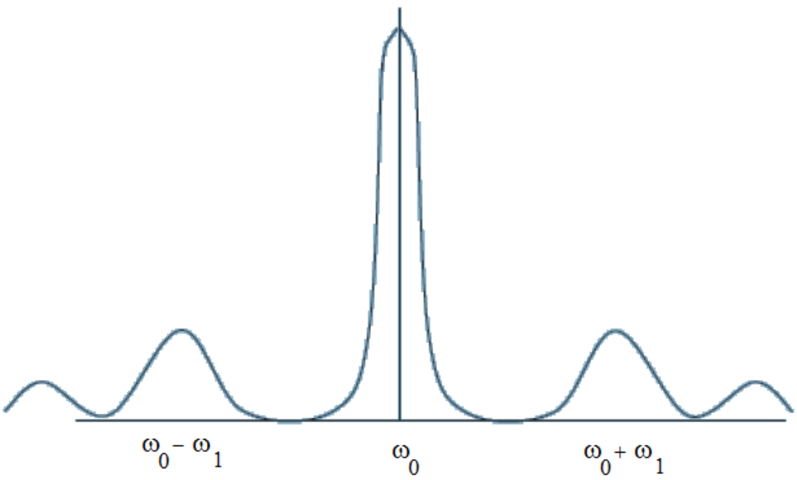

Fig. 5. The wave pick creation and annihilation.

5. APPLICATION TO SONIFICATION

An interesting application of the Burgers' equation is the sonification. The sonification technique is used to explore the tissues images with hardly detectable details. By applying the sonification, new images are obtained with a better visualization of the explored tissue [28, 29]. The approach was exercised on fictive images of fibrotic rat liver samples inspired from a study of effects of ginkgo biloba leaf extract against hepatic toxicity induced by methotrexate in rats in papers [30, 31].

The sonification procedure is: a digital image B seen as a collection of N pixels is subjected to the force $f(t)$ expressed as a sum of the excitation harmonic force $F_p(t)$ and the generation sound force $F_s(t)$. The last force is introduced to build the sonification operator. The behavior of the digital image is described by the Burgers' equation (18). The force F_s is determined from the minimum of the acoustic power radiated by B

$$\frac{\partial W}{\partial F_s} = 0\,,\ \ W = \frac{A}{2} v^T p\,, \tag{37}$$

where v is the sound velocity and p the acoustic pressure vector, A is the area of the rectangular picture, and the subscript T denoted the transpose.

The unknown parameters $P = \{m, k, \omega, \tilde{\phi}, \alpha, \gamma, \lambda\}$ are find by a genetic algorithm which minimizes the objective function $\Upsilon(P_j)$ given by

$$\Upsilon(P) = 3^{-1}\sum_{j=1}^{3} \delta_{1j}^2 + \delta_2^2\,, \tag{38}$$

where δ_{1j} and δ_2 are residuals which must tend to zero

$$\delta_{1j} = \frac{\partial v_j}{\partial x_j} - \frac{\beta_j}{c_0^2} v_j \frac{\partial v_j}{\partial \tau} - \frac{b_j}{2\rho_0 c_0^3} v_j \frac{\partial^2 v_j}{\partial \tau^2}\,,\ \ \delta_2 = \frac{\partial W}{\partial F_s}\,. \tag{39}$$

After sonification, the mapped data is completed and filled with color and geometric lines, through continuity of solutions (27) in adjacent areas, so that the final image may contain details that do not appear in the original image.

We consider now a fictive image of a coagulation necrosis area in the liver, and intentionally hide an area in this image (shown in green in Fig. 6a). Fig. 6b are the image used for sonification. The sonification to this image was successful in the sense that the initially hidden area is recovered (Fig. 6c).

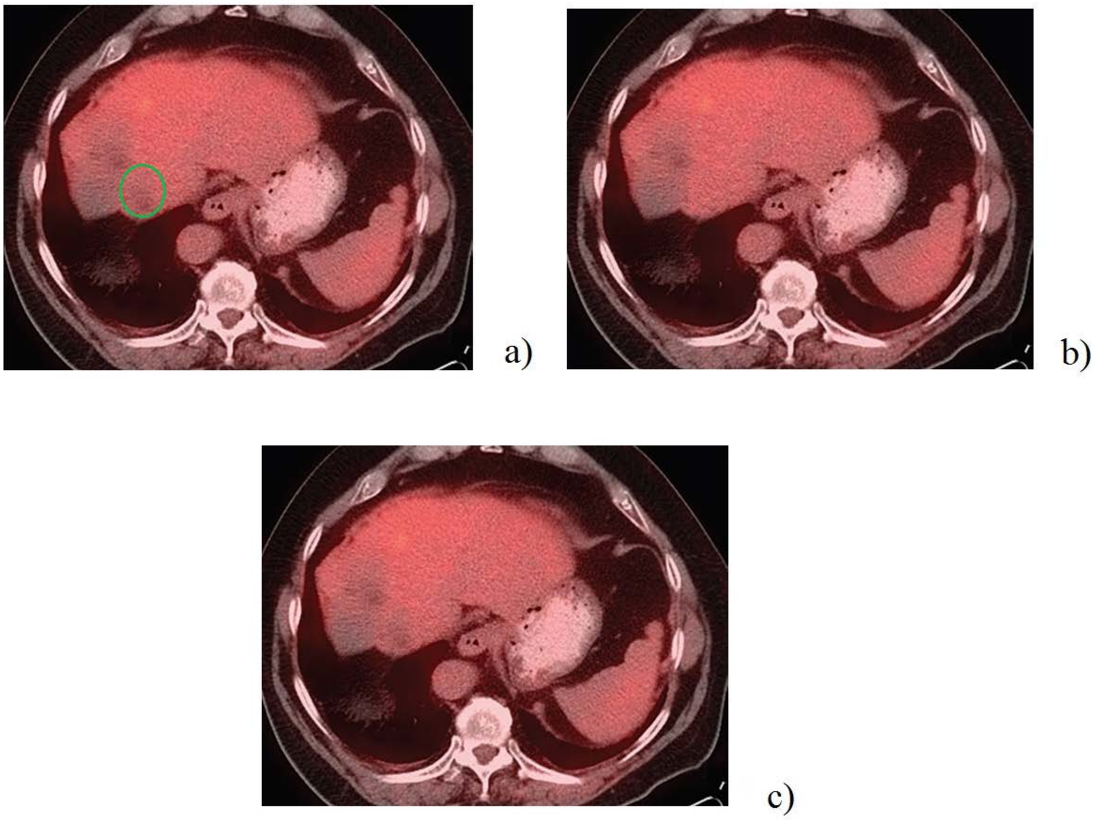

Fig. 6a. The MR image of a liver; b) an image with a hidden area used for sonification; c) the hidden area is found by sonification.

6. CONCLUSIONS

The propagation of sound waves in soft biological tissues (blood, veins, kidney, liver, lung, etc.) is described in this paper by using the Burgers' equation in the range of frequencies 1–10 MHz. The solution of the Burgers' equation are determined by the cnoidal method. Solutions are localized waves that conserve their properties at interaction with other waves. The properties of symmetry of the Burgers' equation, i.e. the space and time translations symmetry, the odd reflection symmetry and the Galilean invariance are the main symmetries of the equation are analysed.

Utility of the Burgers' equation to sonification technique is highlighted next on a medical image used to surgical operations.

For a better application of the Burgers' equation, the acoustic properties of the tissue at ultrasonic frequencies (acoustic velocity, ultrasonic attenuation and factors affecting the acoustic velocity and ultrasonic attenuation - temperature, frequency, anisotropy) and the mechanical properties of the tissue (mass density, elastic moduli, viscoelastic moduli and factors affecting elasticity and viscoelasticity) have to be well known.

Ultrasonic speed, attenuation are functions of frequency. It is observed in the literature that acoustic speed *in vitro* is different between tumors and normal human liver. By comparison with normal liver, ultrasound propagates about 1.5% (± 1%) slower, is attenuated by about 20% (± 30%) less at 3 MHz in the tumor that were measured [33]. It is observed that the ultrasonic velocity decreases with increasing water and fat contents. An increase in the water content is related to the decreasing attenuation, and positive dependences exist between the acoustic characteristics and the fat content [34-36].

The knowledge of the acoustic properties of the tissue at ultrasonic frequencies and the mechanical properties of the tissue is needed in surgical interventions where the surgeon and the robot are working with the same tool-tip, with the goal to minimize the vascular damages and bleeding [37-39]. The inverse problem of extracting the acoustic properties of the tissue at ultrasonic frequencies and the mechanical properties of the tissue from experimental data represents a future work.

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