



RELATIONSHIP BETWEEN THE EQUILIBRIUM AND MOTION EQUATIONS FOR A THIN ELASTIC BEAM

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Abstract The nonlinear equilibrium equations for a thin elastic rod with ends fixed by external forces can be solved exactly. Due to the large displacements, a nonlinear effect sets in. The localized solutions are cnoidal and solitonic waves. The rod deviates from a plane when the torsion exists. In this paper, the conditions for which the motion equations are equivalent to the equilibrium equations, are determined. If these conditions are fulfilled the exact solutions of the motion equations of the thin elastic beam can be determined from the exact solutions of the equilibrium equations.

Key words: thin elastic rod, equilibrium equations, motion equations, cnoidal solutions, solitons.

1. INTRODUCTION

The thin elastic rod or elastica has an important role in the history of vibration theory. Bernoulli introduced in 1755 the superposition principle of a multitude small oscillations at the same time, in the vibrating thin beam. The problem of the bending and twisting of thin rods was solved by Love in 1926 [1]. He obtained the equations of inflexional elastica and non-inflexional elastica, respectively.

Besides the strings and beams there are many physical issues that can be treated as thin elastic beams [2,3]. In biology there exist 1D media such as DNA, RNA and α -helix of protein which can be modeled as thin elastic beams [4-7].

The study or behavior of the thin elastic beam leads to the nonlinear partial differential equations that admit soliton solutions [2]. These solutions are localized disturbances with non-oscillatory motion and the velocity dependent of its amplitude and the hyperbolic secant shape. This contrasts strongly with the linear solutions with velocity independent of the amplitude [8].

These equations have interesting properties: an infinite number of local conserved quantities, an infinite number of exact solutions expressed in terms of the Jacobi elliptic functions or the hyperbolic functions [9-10]. Such equations are integrable and admit simple superposition of solutions.

John Scott Russell discovered the physical soliton in 1834, Russell was observing a boat being drawn along a narrow channel by horses [11].

Details on the soliton theory can be found in [12-16].

In this paper the basic laws of equilibrium and motion for a thin elastic rod are presented.

The exact solutions of the equilibrium equations are expressed by the elliptic and hyperbolic functions. The conditions for which the motion equations are equivalent to the equilibrium equations, are important problem. If these conditions are fulfilled the exact solutions of the motion equations of a thin elastic beam can be determined from the exact solutions of the equilibrium equations.

2. FUNDAMENTAL EQUATIONS

A thin elastic, homogeneous and isotropic beam of length l , and circular cross section of radius $a \ll l$. External forces and couples fix the ends of the beam. The beam deforms in space by bending and torsion. At time $t = 0$ the beam occupies the region $\Omega_0 \subset R^3$, and after motion, the beam occupies the region $\Omega(t)$ at the time t . The motion of the beam between $t = 0$ and $t = t_1$ is known if and only if we know the single valued transformation with continuous partial derivatives with respect to their arguments

$$S(0, t), \quad \forall t \in [0, t_1], \quad (1)$$

which takes a point in Ω_0 at $t = 0$ to a spatial position in $\Omega(t)$ at $t = t_1$.

The position of a point in Ω_0 may be denoted by a Lagrange coordinate system $X \equiv (X, Y, Z)$ and the spatial position of the same point in $\Omega(t)$, by the Euler coordinate system $x \equiv (x, y, z)$.

$$x = f_i(X, t), \quad i = 1, 2, 3. \quad (2)$$

The coordinate along the central line of the beam is s . The orthonormal basis of the Lagrange system of coordinates is (e_1, e_2, e_3) , and the orthonormal basis of the Euler system of coordinates is (d_1, d_2, d_3) . The $\{d_k\}$, $k = 1, 2, 3$ is related to $\{e_k\}$, $k = 1, 2, 3$ by the Euler angles θ, ψ and φ [2, 4, 5]

$$\begin{aligned} d_1 &= (-\sin \psi \sin \varphi + \cos \psi \cos \varphi \cos \theta)e_1 + \\ &\quad + (\cos \psi \sin \varphi + \sin \psi \cos \varphi \cos \theta)e_2 - \sin \theta \cos \varphi e_3, \\ d_2 &= (-\sin \psi \cos \varphi - \cos \psi \sin \varphi \cos \theta)e_1 + \\ &\quad + (\cos \psi \cos \varphi - \sin \psi \sin \varphi \cos \theta)e_2 + \sin \theta \sin \varphi e_3, \\ d_3 &= \sin \theta \cos \psi e_1 + \sin \theta \sin \psi e_2 + \cos \theta e_3. \end{aligned} \quad (3)$$

The motion equations are given by

$$R \times R \ni (s, t) \rightarrow r(s, t), d_1(s, t), d_2(s, t) \in E^3, \quad (4)$$

where $r(s, t)$ is the position vector. The $d_1(s, t), d_2(s, t)$ give the orientation of s in the Euler configuration. The unit tangential vector $d_3(\sin \theta \cos \psi, \sin \theta \sin \psi, \cos \theta)$ along the beam can be expressed as

$$d_3(s, t) = d_1(s, t) \times d_2(s, t). \quad (5)$$

The strains y_1, y_2, y_3 are expressed as

$$r' = y_k d_k, \quad (6)$$

where (\prime) means the partial differentiation with respect to s . The $\{d_k\}$, $k = 1, 2, 3$, are orthonormal, and therefore, the vector u can be defined as

$$d_k' = u \times d_k, \quad (7)$$

with the components with respect to the basis $\{d_k\}$

$$u_k = \frac{1}{2} e_{klm} d'_l \cdot d_m, \quad (8)$$

where e_{klm} are the components of the alternating tensor.

By substitution of (3) into (8) we get

$$\begin{aligned} u_1 &= \theta' \sin \varphi - \psi' \sin \theta \cos \varphi, \\ u_2 &= \theta' \cos \varphi + \psi' \sin \theta \sin \varphi, \\ u_3 &= \varphi' + \psi' \cos \theta. \end{aligned} \quad (9)$$

where u_1 and u_2 represent the components of the curvature κ of the central line associated to the planes (yz) and (xz)

$$\kappa^2 = u_1^2 + u_2^2 = \theta'^2 + \psi'^2 \sin^2 \theta, \quad (10)$$

and u_3 is the torsion τ

$$u_3 = \tau = \varphi' + \psi' \cos \theta. \quad (11)$$

The strains are

$$y_1 = y_2 = 0, \quad y_3 = 1, \quad u_k = 0. \quad (12)$$

The relationship between $r = (x, y, z)$ and d_3 is obtained from (12) and (6)

$$r' = d_3. \quad (13)$$

or

$$r = \int_0^s d_3 ds, \quad (14)$$

and

$$\begin{aligned} x(s) &= \int_0^s \cos \psi \sin \theta ds, \quad y(s) = \int_0^s \sin \psi \sin \theta ds, \\ z(s) &= \int_0^s \cos \theta ds. \end{aligned} \quad (15)$$

We define the vector D of the components $x(L), y(L), z(L)$

$$D = \int_0^l d_3 ds. \quad (16)$$

The elastic energy of the deformed rod U is a sum between the bending and the torsional energies [2]

$$U = \frac{A}{2} \int_0^l \kappa^2 ds + \frac{C}{2} \int_0^l \tau^2 ds. \quad (17)$$

The constants A and C are the bending stiffness and respectively the torsional stiffness related to the elastic constants λ , μ by

$$A = \frac{1}{4} \pi a^4 E, \quad C = \frac{1}{2} \pi a^4 \mu, \quad (18)$$

where $E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$ is the elastic modulus, and a is the radius of the cross section of the beam.

From (10) and (11) we obtain

$$U = \frac{A}{2} \int_0^l (\theta'^2 + \psi'^2 \sin^2 \theta) ds + \frac{C}{2} \int_0^l (\varphi' + \psi' \cos \theta)^2 ds. \quad (19)$$

The equilibrium equations are obtained from variation of the elastic energy U with respect to θ , φ and ψ .

The exact static equilibrium equations of the thin elastic beam with the ends fixed by the external force $F = -\lambda$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ are

$$A(\psi'^2 \sin \theta \cos \theta - \theta'') - C(\varphi' + \psi' \cos \theta)\psi' \sin \theta + \lambda_1 \cos \theta \cos \psi + \lambda_2 \cos \theta \sin \psi - \lambda_3 \sin \theta = 0, \quad (20)$$

$$\frac{\partial}{\partial s} [A\psi' \sin^2 \theta + C(\varphi' + \psi' \cos \theta) \cos \theta] - \lambda_1 \sin \theta \sin \psi + \lambda_2 \sin \theta \cos \psi = 0, \quad (21)$$

$$\frac{\partial}{\partial s} [-C(\varphi' + \psi' \cos \theta)] = 0. \quad (22)$$

The end couples at $s=0$ and $s=1$ are $M_i(0)$, $i=1,2,3$, and $M_i(l)$, $i=1,2,3$,

$$M_1(s) = A\theta', \quad (23)$$

$$M_2(s) = A\psi' \sin^2 \theta + C(\psi' \cos \theta + \varphi') \cos \theta, \quad (24)$$

$$M_3(s) = C(\varphi' \cos \theta + \psi'). \quad (25)$$

We introduce next the beam inertia

$$R \ni s \rightarrow (\rho_0 A_0)(s), (\rho_0 I_1)(s), (\rho_0 I_2)(s) \in (0, \infty), \quad (26)$$

where $(\rho_0 A_0)$ is the mass density per unit length, A_0 the area of the cross section, $(\rho_0 I_1)$ the principal mass moment of inertia with respect the perpendicular axis to the central axis, and $(\rho_0 I_2)$ the principal mass moment of inertia around the central axis. We have

$$\rho = A_0 \rho_0 = \pi a^2 \rho_0, \quad k_1 = I_1 \rho_0 = \frac{\pi a^4}{4} \rho_0, \quad k_2 = I_2 \rho_0 = \frac{\pi a^4}{2} \rho_0, \quad (27)$$

where ρ_0 is the mass density per unit volume, and I_1, I_2 moments of inertia with respect to perpendicular axis to the central axis, and respectively around the central axis. The kinetic energy K is a sum between the translational energy K_1 , the rotational energy of the tangential vector K_2 , and the rotational energy around the central axis K_3 [2]

$$K = K_1 + K_2 + K_3, \quad (28)$$

with

$$K_1 = \frac{\rho}{2} \int_0^l \dot{r}^2 ds, \quad (29)$$

$$K_2 = \frac{k_1}{2} \int_0^l \dot{d}_3^2 ds = \frac{k_1}{2} \int_0^l (\Omega_1^2 + \Omega_2^2) ds, \quad (30)$$

$$K_3 = \frac{k_2}{2} \int_0^l \Omega_3^2 ds, \quad (31)$$

where dot is differentiation with respect to time, and $\Omega(\Omega_1, \Omega_2, \Omega_3)$ is angular velocity of rotation vector

$$\begin{aligned} \Omega_1 &= -\dot{\psi} \sin \theta \cos \varphi + \dot{\theta} \sin \varphi, \\ \Omega_2 &= \dot{\psi} \sin \theta \sin \varphi + \dot{\theta} \cos \varphi, \\ \Omega_3 &= \dot{\psi} \cos \theta + \dot{\varphi}. \end{aligned} \quad (32)$$

The (32) are analogous to (9). Using (7) K_2 and K_3 take the form

$$K_2 = \frac{k_1}{2} \int_0^l (\dot{\psi}^2 \sin^2 \theta + \dot{\theta}^2) ds, \quad (33)$$

$$K_3 = \frac{k_2}{2} \int_0^l (\dot{\psi} \cos \theta + \dot{\varphi})^2 ds. \quad (34)$$

The motion equations of the beam fixed by $F = -\lambda$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ are

$$-\rho \ddot{r} - \lambda' = 0, \quad (35)$$

$$\begin{aligned} &k_1 (\dot{\psi}^2 \sin \theta \cos \theta - \ddot{\theta}) - k_2 (\dot{\varphi} + \dot{\psi} \cos \theta) \dot{\psi} \sin \theta - \\ &- A (\psi'^2 \sin \theta \cos \theta - \theta'') + C (\varphi' + \psi' \cos \theta) \psi' \sin \theta - \\ &- \lambda_1 \cos \theta \cos \psi - \lambda_2 \cos \theta \sin \psi + \lambda_3 \sin \theta = 0, \end{aligned} \quad (36)$$

$$\begin{aligned} &-\frac{\partial}{\partial t} \{k_1 \dot{\psi} \sin^2 \theta + k_2 (\dot{\varphi} + \dot{\psi} \cos \theta) \cos \theta\} + \\ &+\frac{\partial}{\partial s} \{A \psi'^2 \sin^2 \theta + C (\varphi' + \psi' \cos \theta) \cos \theta\} + \\ &+ \lambda_1 \sin \theta \sin \psi - \lambda_2 \sin \theta \cos \psi = 0, \end{aligned} \quad (37)$$

$$-k_2 \frac{\partial}{\partial t} (\dot{\varphi} + \dot{\psi} \cos \theta) + C \frac{\partial}{\partial s} (\varphi' + \psi' \cos \theta) = 0. \quad (38)$$

3. EQUIVALENCE BETWEEN THE EQUILIBRIUM AND MOTION EQUATIONS

Given λ, θ, ψ and φ as functions only of the variable $\xi = s - vt$ and suppose that [2]

$$\lambda = \zeta d_3 + (0, 0, \lambda_3), \quad (39)$$

with $\zeta = -\rho v^2$ and $d_3 = (\sin \theta \cos \psi, \sin \theta \sin \psi, \cos \theta)$.

We are interested to determine next the conditions for which the motion equations (35)–(38) are equivalent to the equilibrium equations (20)–(22).

$$A - k_1 v^2 \rightarrow A, \quad C - k_2 v^2 \rightarrow C, \quad \xi \rightarrow s. \quad (40)$$

The motion equations (35)–(38) can be written as

$$-\rho v^2 r'' = \lambda', \quad (41)$$

$$\begin{aligned} & k_1 v^2 (\psi'^2 \sin \theta \cos \theta - \theta'') - k_2 v^2 (\varphi' + \psi' \cos \theta) \psi' \sin \theta - \\ & - A (\psi'^2 \sin \theta \cos \theta - \theta'') + C (\varphi' + \psi' \cos \theta) \psi' \sin \theta - \\ & - \lambda_1 \cos \theta \cos \psi - \lambda_2 \cos \theta \sin \psi + \lambda_3 \sin \theta = 0, \end{aligned} \quad (42)$$

$$\begin{aligned} & -\frac{\partial}{\partial \xi} \{-v k_1 \psi' \sin^2 \theta - v k_2 (\varphi' + \psi' \cos \theta) \cos \theta + \\ & + A \psi'^2 \sin^2 \theta + C (\varphi' + \psi' \cos \theta) \cos \theta\} + \lambda_1 \sin \theta \sin \psi - \\ & - \lambda_2 \sin \theta \cos \psi = 0, \end{aligned} \quad (43)$$

$$-k_2 \frac{\partial}{\partial \xi} [-v \varphi' - v \psi' \cos \theta + C (\varphi' + \psi' \cos \theta)] = 0, \quad (44)$$

when we refer them to a new variable $\xi = s - vt$, or

$$-\rho v^2 r' = \lambda + \hat{c}, \quad (45)$$

$$\begin{aligned} & (A - k_1 v^2) (\psi'^2 \sin \theta \cos \theta - \theta'') - \\ & - (C - k_2 v^2) (\varphi' + \psi' \cos \theta) \psi' \sin \theta + \\ & + \lambda_1 \cos \theta \cos \psi + \lambda_2 \cos \theta \sin \psi - \lambda_3 \sin \theta = 0, \end{aligned} \quad (46)$$

$$\begin{aligned} & (A - k_1 v^2) (\psi'' \sin \theta + 2\psi' \theta' \cos \theta) - \\ & (C - k_2 v^2) \theta' (\varphi' + \psi' \cos \theta) + \\ & + \lambda_1 \sin \psi - \lambda_2 \cos \psi = 0, \end{aligned} \quad (47)$$

$$\varphi'(s) + \psi'(s) \cos \theta(s) = \tau, \quad (48)$$

with \hat{c} an integration constant.

The equation (46) becomes

$$\lambda = (-\rho v^2 \sin \theta \cos \psi - \hat{c}_1, \quad -\rho v^2 \sin \theta \sin \psi - \hat{c}_2, \quad -\rho v^2 \cos \theta - \hat{c}_3). \quad (49)$$

From (39) we have

$$\lambda = (\zeta \sin \theta \cos \psi, \zeta \sin \theta \sin \psi, \zeta \cos \theta + \lambda_3). \quad (50)$$

The equations (49) and (50) are equivalent if

$$\zeta = -\rho v^2, \hat{c} = (0, 0, -\lambda_3). \quad (51)$$

In these conditions, the motion equations (35)–(38) become

$$(A - k_1 v^2)(\psi'^2 \sin \theta \cos \theta - \theta'') - (C - k_2 v^2)(\varphi' + \psi' \cos \theta)\psi' \sin \theta - \lambda_3 \sin \theta = 0, \quad (51)$$

$$(A - k_1 v^2)(\psi'' \sin \theta + 2\psi' \theta' \cos \theta) - (C - k_2 v^2)\theta'(\varphi' + \psi' \cos \theta) = 0, \quad (52)$$

$$\varphi'(s) + \psi'(s) \cos \theta(s) = \tau. \quad (53)$$

Hence, the motion equations (51)–(53) coincide with equilibrium equations (35)–(38) if

$$A - k_1 v^2 \rightarrow A, \quad C - k_2 v^2 \rightarrow C, \quad \xi \rightarrow s. \quad (54)$$

The conditions (54) make possible to write the exact solutions of the motion equations of the thin elastic beam the solutions of the equilibrium equations.

The equilibrium equations (35)–(38) have a unique solution for the Euler angles

$$u = u_2 - (u_2 - u_3) \operatorname{cn}^2 \left(\sqrt{\frac{|\lambda_3|}{2A}} (u_1 - u_3)(s - s_3), m \right) = u_2 - (u_2 - u_3) \operatorname{cn}^2 [w(s - s_3), m], \quad (55)$$

with

$$m = \frac{u_2 - u_3}{u_1 - u_3}, \quad w = \sqrt{\frac{|\lambda_3|}{2A}} (u_1 - u_3),$$

$$\psi = \frac{1}{4A^2 w^2} \left\{ -\frac{\beta + C\tau}{1 - u_3} \Pi \left[w(s - s_3), \frac{u_2 - u_3}{1 - u_3}, m \right] - \frac{\beta - C\tau}{1 + u_3} \Pi \left[w(s - s_3), \frac{u_2 - u_3}{1 + u_3}, m \right] \right\}, \quad (56)$$

$$\varphi = -\frac{\tau(C - A)}{A} s + \frac{1}{4A^2 w^2} \left\{ \frac{\beta + C\tau}{1 - u_3} \Pi \left[w(s - s_3), \frac{u_2 - u_3}{1 - u_3}, m \right] - \frac{\beta - C\tau}{1 + u_3} \Pi \left[w(s - s_3), \frac{u_2 - u_3}{1 + u_3}, m \right] \right\}, \quad (57)$$

where $\Pi(x, z, m)$ is the normal elliptic integral of the third kind

$$\Pi(x, z, m) = \int_0^x \frac{dy}{1 - z \operatorname{sn}^2(y, m)}. \quad (58)$$

In (55)–(58), $u_3 < u_2 < u_1$ are distinct and real roots of the cubic equation

$$au^3 + bu^2 - au + c = 0, \quad (59)$$

The graphical representation of the solution u for different values of m are shown in Fig.1.

If (54) are fulfilled, the motion equations (35)–(38) have a unique solution for the Euler angles

$$\begin{aligned} u &= u_2 - (u_2 - u_3) \operatorname{cn}^2 \left(\sqrt{\frac{|\lambda_3|}{2A}} (u_1 - u_3) (\xi - \xi_3), m \right) = \\ &= u_2 - (u_2 - u_3) \operatorname{cn}^2 [w(\xi - \xi_3), m], \end{aligned} \quad (60)$$

with

$$m = \frac{u_2 - u_3}{u_1 - u_3}, \quad w = \sqrt{\frac{|\lambda_3|}{2A}} (u_1 - u_3),$$

$$\begin{aligned} \psi &= \frac{1}{4(A - k_1 v^2)^2 w^2} \left\{ -\frac{\beta + (C - k_2 v^2) \tau}{1 - u_3} \Pi[w(\xi - \xi_3), \frac{u_2 - u_3}{1 - u_3}, m] - \right. \\ &\quad \left. - \frac{\beta - (C - k_2 v^2) \tau}{1 + u_3} \Pi[w(\xi - \xi_3), \frac{u_2 - u_3}{1 + u_3}, m] \right\}, \end{aligned} \quad (61)$$

$$\begin{aligned} \varphi &= -\frac{\tau [C - A - (k_2 + k_1) v^2]}{A - k_1 v^2} \xi + \frac{1}{4(A - k_1 v^2)^2 w^2} \left\{ \frac{\beta + (C - k_2 v^2) \tau}{1 - u_3} \times \right. \\ &\quad \left. \times \Pi[w(\xi - \xi_3), \frac{u_2 - u_3}{1 - u_3}, m] - \frac{\beta - (C - k_2 v^2) \tau}{1 + u_3} \Pi[w(\xi - \xi_3), \frac{u_2 - u_3}{1 + u_3}, m] \right\}, \end{aligned} \quad (62)$$

where $\Pi(x, z, m)$ is the normal elliptic integral of the third kind, and $u_3 < u_2 < u_1$ the distinct and real roots of the cubic equation (59).

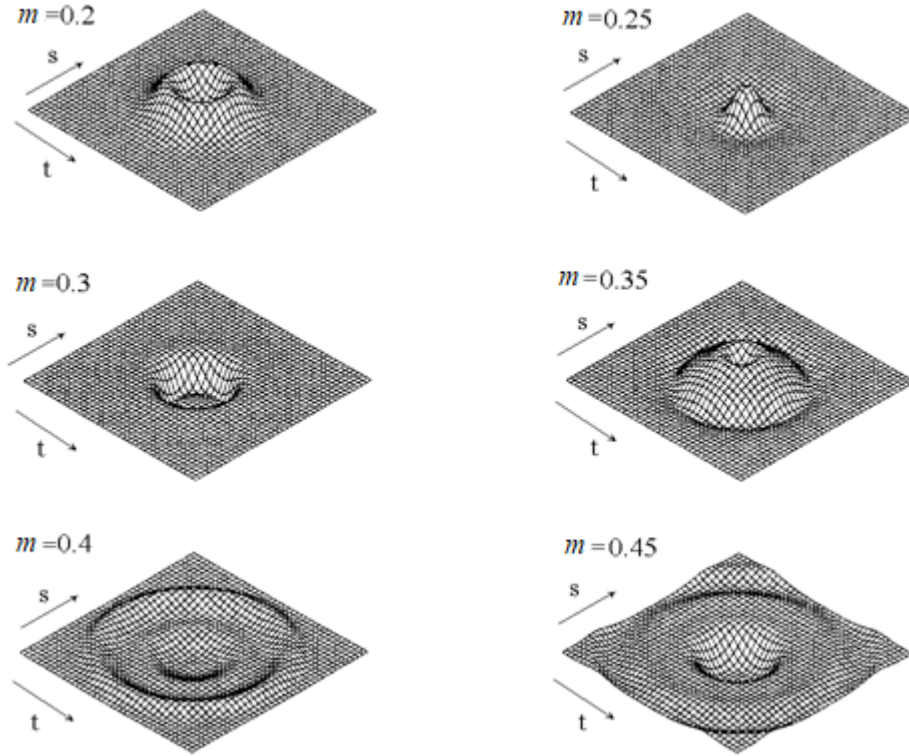


Fig.1. The solution u for different m .

Both solutions are expressed with respect to cnoidal functions. If we compare the integral

$$v = \int_0^\varphi \frac{d\theta}{(1 - m \sin^2 \theta)^{1/2}}, \quad (63)$$

where $0 \leq m \leq 1$, to the elementary integral

$$w = \int_0^\varphi \frac{dt}{(1 - t^2)^{1/2}}, \quad (64)$$

and observed that (64) defines the inverse of the trigonometric function \sin if we use the notations $t = \sin \theta$ and $\psi = \sin w$. Therefore, the cnoidal functions can be defined as a pair of inverse functions from (63)

$$\operatorname{sn} v = \sin \varphi, \quad \operatorname{cn} v = \cos \varphi. \quad (65)$$

The angle φ is called the amplitude $\varphi = \operatorname{am} u$. We also define the Jacobi elliptic function $\operatorname{dn} v = (1 - m \sin^2 \varphi)^{1/2}$. For $m = 0$, we have

$$\begin{aligned} v &= \varphi, \quad \operatorname{cn}(v, 0) = \cos \varphi = \cos v, \\ v &= \varphi \operatorname{sn}(v, 0) = \sin \varphi = \sin v, \quad \operatorname{dn}(v, 0) = 1, \end{aligned} \quad (66)$$

and for $m = 1$ we have the solitons

$$\begin{aligned} v &= \operatorname{arcsech}(\cos \varphi), \quad \operatorname{cn}(v, 1) = \operatorname{sech} v, \\ \operatorname{sn}(v, 1) &= \tanh v, \quad \operatorname{dn}(v, 1) = \operatorname{sech} v. \end{aligned} \quad (67)$$

The soliton solutions are localized waves that conserve their properties even after interaction among them, and then act somewhat like particles. Fig.2 shows some features of solitons. Fig.2a and b presents the collision between two soliton waves, and Fig 2c and d presents the takeover of two soliton waves. If consider two soliton waves with different velocities $v_1 > v_2$, the faster wave overtakes the slow soliton, as shown in Fig. 3. A larger soliton travels with a high velocity and it overtakes the smaller soliton, and after the collision they reappear without changing their shape and identity [17-22].

The solitons were regarded as a curiosity until the 1960s, when the using of computers help the study of solitons propagation. The discovery of solitons started with the analysis of nonlinear partial differential equations, such as the work of Boussinesq and Rayleigh, independently, in the 1870s [16]. They explained mathematically the Russell discovery and derived the water motion equation in shallow channels which confirmed the existence of solitons.

We consider next the case $u_1 = u_2 = -1$, $u_3 = 1$, of the roots of the cubic equation (59). The Euler angles are determined from the equilibrium equations (55)–(57)

$$u(s) = -1 + 2 \frac{|\lambda_3|}{A} \operatorname{sech}^2 \sqrt{\frac{|\lambda_3|}{A}} s, \quad (68)$$

$$\psi = \frac{-C\tau s}{2A} + \arctan \left(\frac{4A}{C\tau} \tanh \left(-\sqrt{\frac{|\lambda_3|}{A}} s \right) \right), \quad (69)$$

$$\varphi = \frac{\tau(2A - C)s}{2A} + \arctan \left(\frac{4A}{C\tau} \tanh \sqrt{\frac{|\lambda_3|}{A}} s \right). \quad (70)$$

Fig. 4 displays the shapes of elastica for $\tau = 0$, while Fig.5 presents the shapes of elastica for $\tau \neq 0$. These shapes are similar to the shapes of elastica found by Love in 1926 [1].

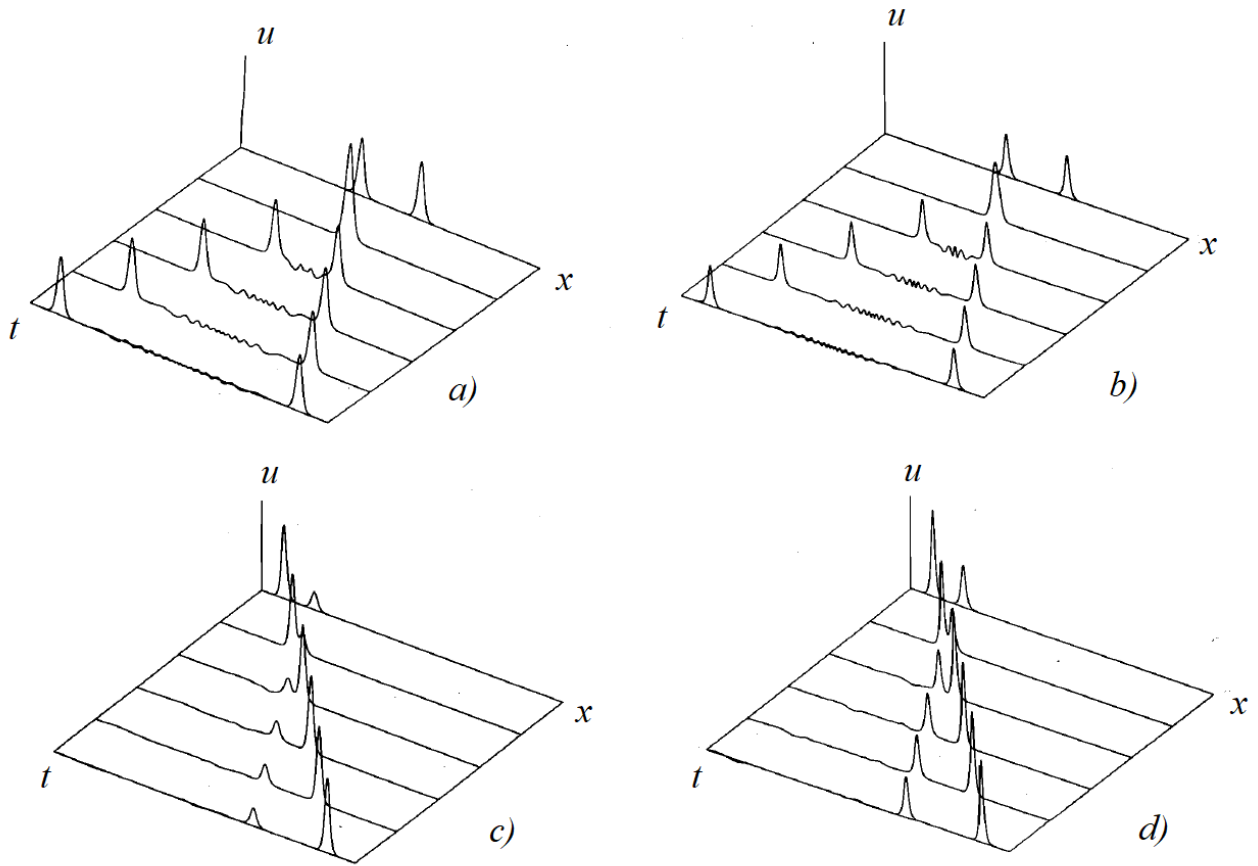
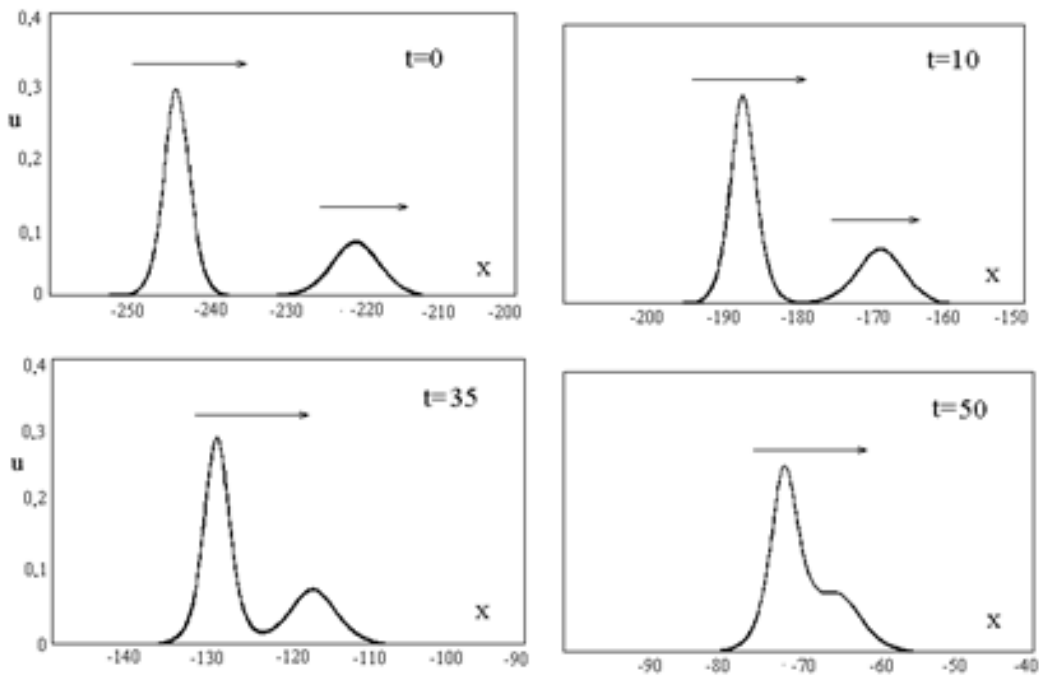


Fig.2. Some features of solitons, a), b) collision of two soliton waves and c)d) takeover of two solitons.



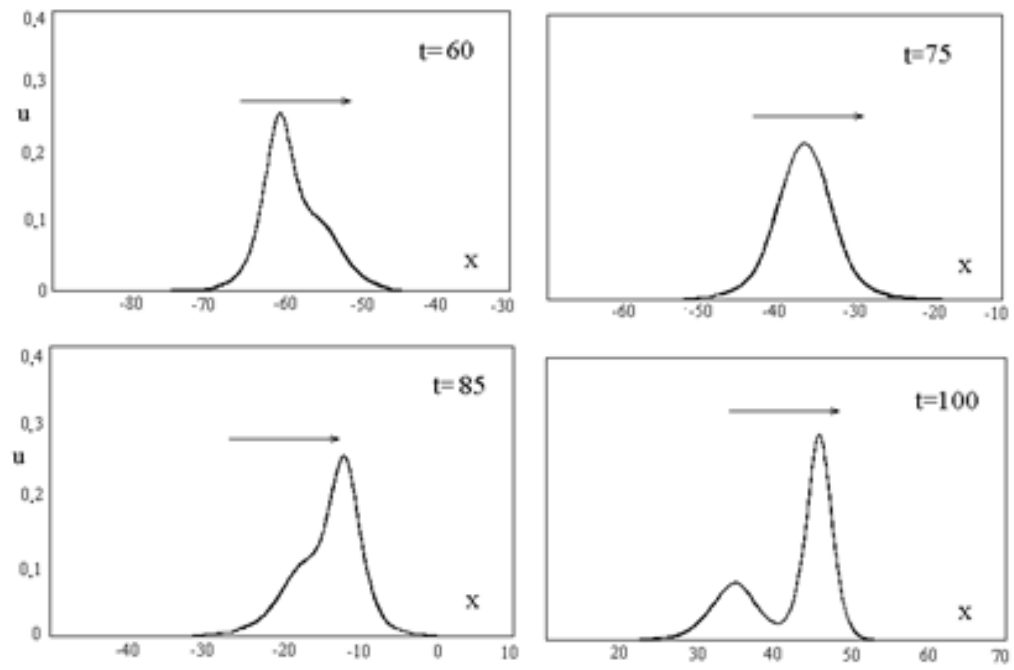


Fig.3. The interaction of two solitons running in the same direction.

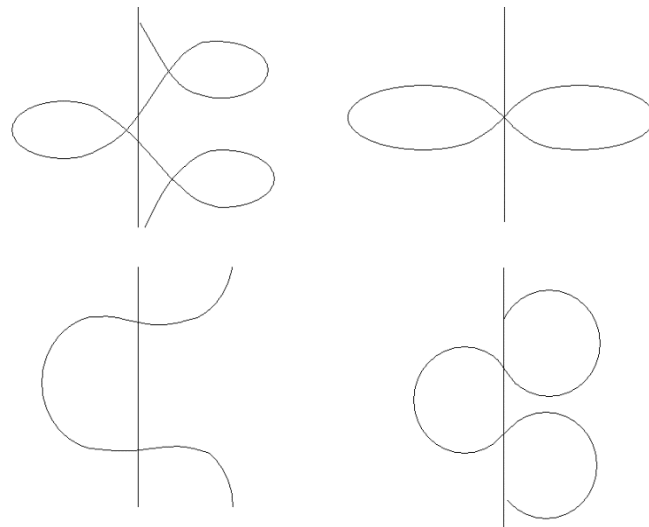


Fig. 4. Shapes of elastica of Love for $\tau = 0$.



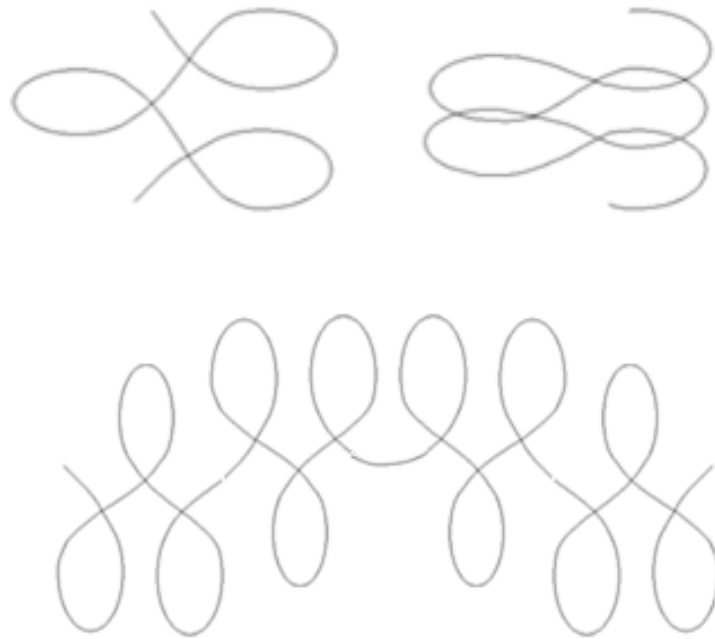


Fig. 5. Shapes of elastica of Love for $\tau \neq 0$.

4. CONCLUSIONS

The equilibrium and motion equations for a thin elastic rod with ends fixed by external forces are studied in this paper. The conditions for which the motion equations are equivalent to the equilibrium equations, are determined. If these conditions are fulfilled the exact solutions of the motion equations of the thin elastic beam can be determined from the exact solutions of the equilibrium equations.

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